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TOPICAL REVIEW

On the geometric approach to the motion of inertial mechanical systems

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Abstract
According to the principle of least action, the spatially periodic motions of one-dimensional mechanical systems with no external forces are described in the Lagrangian formalism by geodesics on a manifold-configuration space, the group $\mathcal{D}$ of smooth orientation-preserving diffeomorphisms of the circle. The periodic inviscid Burgers equation is the geodesic equation on $\mathcal{D}$ with the L^2 right-invariant metric. However, the exponential map for this right-invariant metric is not a C^1 local diffeomorphism and the geometric structure is therefore deficient. On the other hand, the geodesic equation on $\mathcal{D}$ for the H^1 right-invariant metric is also a re-expression of a model in mathematical physics. We show that in this case the exponential map is a C^1 local diffeomorphism and that if two diffeomorphisms are sufficiently close on $\mathcal{D}$, they can be joined by a unique length-minimizing geodesic—a state of the system is transformed to another nearby state by going through a uniquely determined flow that minimizes the energy. We also analyse for both metrics the breakdown of the geodesic flow.

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1. Introduction

Motions of mechanical systems with no external forces are described in the Lagrangian formalism by paths on a configuration space G that is a Lie group. The velocity phase space is the tangent bundle TG of G. Let $\mathcal{G}$ be the Lie algebra of G—the tangent space at the neutral element of the group. For a nondegenerated inner product $\langle\cdot,\cdot\rangle$, the quantity $\frac{1}{2}\langle v, v\rangle$, $v \in \mathcal{G}$, is called the kinetic energy K. We can extend K by right or left translation[3] to a

[3] In general, the Lagrangian is a scalar function $L : TG \to \mathbb{R}$ so that constancy under particular transformation of its arguments is the only sort of symmetry to which it can be subject. This explains the preferred choice of right- or left-invariance.

right- or left-invariant Lagrangian $L : TG \to \mathbb{R}$ in order to define a 'natural Lagrangian system' on G [2]. The action along a path $g(t)$, $a \leqslant t \leqslant b$, in G is defined as $\int_a^b L(g, g_t)\,\mathrm{d}t$. The action principle (cf [41]) states that the equation of motion is the equation satisfied by an extremal (a critical point) of the action in the space of curves on G, the paths $g(t)$ over which we are extremizing satisfying the fixed end conditions $g(a) = g_0$ and $g(b) = g_1$. In many cases (cf [2]), the paths described by the motion of a mechanical system are not only extremals but also (local) minimal values of the action functional—the principle of least action holds. Observe that if $g(t)$, $a \leqslant t \leqslant b$, is a C^1-regular path (i.e. $g_t \neq 0$ on $[a, b]$) joining $g(a) = g_0$ to $g(b) = g_1$, the action $\mathfrak{a}(g) = \frac{1}{2}\int_a^b \langle g_t, g_t\rangle\,\mathrm{d}t$ depends on the parametrization of the path. On the other hand, the length $\mathfrak{l}(g) = \int_a^b \langle g_t, g_t\rangle^{\frac{1}{2}}\,\mathrm{d}t$ does not depend on the parametrization and $\mathfrak{l}^2(g) \leqslant 2(b-a)\mathfrak{a}(g)$, with equality if and only if $\langle g_t, g_t\rangle$ is constant on $[a, b]$. From here we infer that the (local) minimum of the action is realized by the curve of minimal length joining g_0 to g_1. In conclusion, for the principle of least action to hold, it is necessary that the equation of motion is the geodesic equation on the configuration manifold.

The configuration space of a rigid body[4] fixed at its centre of mass is the group $SO(3)$ of rotations of $\mathbb{R}^3$. An element g of the group corresponds to a position of the body obtained by the motion g from some arbitrarily chosen initial state (corresponding to the identity element of the group) and a rotation velocity g_t of the body is a vector in the tangent space T_gG. The kinetic energy of a body is determined by the vector of angular velocity in the body (obtained by carrying the tangent vector to $\mathcal{G}$, the tangent space at the identity, by left translation) and does not depend on the position of the body in the space. Therefore, the kinetic energy gives a left-invariant Riemannian metric on the group. By the principle of least action [2] the motion of a rigid body with no external forces is geodesic in $SO(3)$ with this left-invariant metric.

The motion of a system in continuum mechanics is described by a path of diffeomorphisms $\varphi(t, \cdot)$ of the ambient space. The knowledge of $\varphi(t, \cdot)$ gives the configuration of the particles at time t. The material velocity field is defined by $(t, x) \mapsto \varphi_t(t, x)$ while the spatial velocity field is given by $u(t, y) = \varphi_t(t, x)$ where $y = \varphi(t, x)$, i.e. $u(t, \cdot) = \varphi_t \circ \varphi^{-1}$. In terms of u we have the spatial or Eulerian description (from the viewpoint of a fixed observer) while in terms of (φ, φ_t) we have the material or Lagrangian description (the motion as seen from one of the particles—the observer follows the particle). Note the following right-invariance property: if we replace the path $t \mapsto \varphi(t)$ by $t \mapsto \varphi(t) \circ \eta$ for a fixed time-independent $\eta \in \mathcal{D}$, then the spatial velocity $u = \varphi_t \circ \varphi^{-1}$ is unchanged. This suggests the choice of right-invariance rather than left-invariance. In the case of a perfect fluid (nonviscous, homogeneous and incompressible) moving in a bounded smooth domain $M \subset \mathbb{R}^k$, $k = 2, 3$, the configuration space is the group of all volume-preserving diffeomorphisms of M. Arnold [1] observed that the kinetic energy of the fluid, $\frac{1}{2}\int_M |u(t, x)|^2\,\mathrm{d}x$, is invariant with respect to right translations. The invariance of the kinetic energy with respect to right translations is due to incompressibility (the diffeomorphisms are volume preserving), as one can see from a simple change of variables. The obtained geodesic equation is the Euler equation of hydrodynamics [1].

In this review we consider the one-dimensional compressible analogue of the description of the Euler equation for a perfect fluid in two and three dimensions by means of geodesics on the group of volume-preserving diffeomorphisms, a description established by Arnold [1] and placed on a rigorous foundation by Ebin and Marsden [22]. The group $\mathcal{D}$ of smooth orientation-preserving diffeomorphisms of the circle $\mathbb{S}$ (the real numbers modulo 1)

[4] A rigid body is a system of point masses constrained by the fact that the distance between points is constant [2].

represents the configuration space for the spatially periodic motion of inertial one-dimensional mechanical systems.

The choice of the L^2 inner product on each tangent space does not provide us with a right-invariant metric in the one-dimensional compressible case—incompressibility in one dimension would force the diffeomorphisms to be linear. We are therefore led to define an inner product on the tangent space at the identity and produce a right-invariant metric by transporting this inner product to all tangent spaces of $\mathcal{D}$ by means of right translations.

For the L^2 right-invariant metric one obtains the inviscid Burgers equation as the geodesic equation on $\mathcal{D}$,

$$u_t + 3uu_x = 0. \tag{1.1}$$

The geometric approach is meaningful if we are able to use some methods that have been developed in finite-dimensional Riemannian geometry. Unfortunately, as we shall see in section 3, the Riemannian exponential map is not a C^1 local diffeomorphism in the case of the L^2 right-invariant metric.

This raises the natural question whether another right-invariant metric may lead to meaningful results. In view of this, we study the geodesic flow on $\mathcal{D}$ endowed with the H^1 right-invariant metric[5]. The choice of this metric is motivated by the fact that the corresponding geodesic equation is a re-expression of a model arising both in shallow water theory [8] and in elasticity [18],

$$u_t + uu_x + \partial_x \left(1 - \partial_x^2\right)^{-1} \left(u^2 + \tfrac{1}{2}u_x^2\right) = 0. \tag{1.2}$$

In any direction at a given point of $\mathcal{D}$ there exists a smooth geodesic on $\mathcal{D}$. We show that the Riemannian exponential map of the H^1 right-invariant metric is a C^1 local diffeomorphism. We also prove that with the H^1 right-invariant metric $\mathcal{D}$ is not geodesically complete and we analyse the breakdown of the geodesic flow. Finally, we show that if two diffeomorphisms are sufficiently close on $\mathcal{D}$, they can be joined by a unique length-minimizing geodesic of the H^1 right-invariant metric within $\mathcal{D}$. This can be reformulated as a variational problem in the family of smooth diffeomorphisms of the circle and illustrates the power of the geometric approach. Intuitively, it says that a state of the system is transformed to another nearby state by going through a uniquely determined flow of (1.2) that minimizes the energy.

2. Right-invariant metrics on $\mathcal{D}$

In this section we present the manifold and Lie group structure of $\mathcal{D}$, the group of orientation-preserving C^∞ diffeomorphisms of the circle, and we discuss the endowment of $\mathcal{D}$ with a Riemannian structure.

2.1. The diffeomorphism group

$\mathcal{D}$ is a connected manifold modelled on the Fréchet space $C^\infty(\mathbb{S})$ of smooth maps of the circle (the family of real smooth maps on $\mathbb{R}$ of period 1), cf [26]. Recall that a Fréchet space is a complete metrizable topological vector space, its topology being defined by a countable collection of seminorms $\{\|\cdot\|_n\}$: a sequence $u_j \to u$ if and only if for all $n \geqslant 1$ we have $\|u_j - u\|_n \to 0$ as $j \to \infty$. On $C^\infty(\mathbb{S})$ we consider the seminorms to be the $H^k(\mathbb{S})$-norms with $k \geqslant 0$. If F_1, F_2 are Fréchet spaces, $U \subset F_1$ is open and $f : U \subset F_1 \to F_2$ is a continuous map, the derivative of f at $u \in U$ in the direction $v \in F_1$ is defined by

[5] $H^k(\mathbb{S})$, $k \in \mathbb{N}$, stands for the Sobolev space of functions with distributional derivatives up to order k having finite $L^2(\mathbb{S})$ norm.

$Df(u)v = \lim_{t\to 0} \frac{f(u+tv)-f(u)}{t}$. We say that f is C^1 on U if the limit exists for all $u \in U$, $v \in F_1$ and if $Df : U \times F_1 \to F_2$ is continuous[6]. Higher derivatives are defined as derivatives of the lower ones.

The composition and the inverse are both smooth maps from $\mathcal{D} \times \mathcal{D} \to \mathcal{D}$, respectively $\mathcal{D} \to \mathcal{D}$, so that the group $\mathcal{D}$ is a Lie group [26]. The Lie algebra $\mathcal{G}$ of $\mathcal{D}$ is the tangent space to $\mathcal{D}$ at the identity, $T_{Id}\mathcal{D} \equiv C^\infty(\mathbb{S})$, with the bracket

$$[u, v] = -(u_x v - u v_x) \qquad u, v \in \mathcal{G}.$$

Each vector field v on $\mathbb{S}$ (equivalently, each $v \in T_{Id}\mathcal{D}$) gives rise to a one-parameter group of diffeomorphisms $\{\eta(t,\cdot)\}$ obtained as solutions of the differential equation

$$\eta_t = v(\eta) \quad \text{in} \quad C^\infty(\mathbb{S}) \tag{2.1}$$

with initial condition $\eta(0) = Id \in \mathcal{D}$. On the other hand, each one-parameter subgroup $t \mapsto \eta(t) \in \mathcal{D}$ is uniquely determined by its infinitesimal generator $v = \frac{\partial}{\partial t}\eta(t)\Big|_{t=0} \in T_{Id}\mathcal{D}$, the limit being considered in the $C^\infty(\mathbb{S})$ topology. Evaluating the flow $t \mapsto \eta(t,\cdot)$ determined by (2.1) at $t = 1$ we obtain a diffeomorphism $\exp_L(v)$. The diffeomorphism $\eta(t,\cdot)$ is given explicitly by $\eta(t, x_0) = x(t, x_0)$, $x_0 \in \mathbb{S}$, where $x(t, x_0)$ is the unique global solution of the ordinary differential equation $\frac{\mathrm{d}x}{\mathrm{d}t} = v(x)$ with data $x(0) = x_0$, cf [35]. The map $v \to \exp_L(v)$, called the Lie-group exponential map, is a smooth map of the Lie algebra to the Lie group. Although the derivative of $\exp_L$ at the zero vector field is the identity, $\exp_L$ is not locally surjective [35] so that the Lie-group exponential map cannot be used as a local chart on $\mathcal{D}$. This failure is possible since the inverse function theorem does not necessarily hold in Fréchet spaces [26]. Note the contrast with the case of finite-dimensional Lie groups where the map $\exp_L$ is always a local diffeomorphism from the Lie algebra to the Lie group [37].

Let $\mathcal{F}(\mathcal{D})$ be the ring of smooth real-valued functions defined on $\mathcal{D}$ and $\mathcal{X}(\mathcal{D})$ be the $\mathcal{F}(\mathcal{D})$-module of smooth vector fields on $\mathcal{D}$. For $X \in \mathcal{X}(\mathcal{D})$ and $f \in \mathcal{F}(\mathcal{D})$, we define in a local chart the Lie derivative $\mathcal{L}_X f$ as

$$\mathcal{L}_X f(\varphi) = \lim_{h\to 0} \frac{f(\varphi + hX(\varphi)) - f(\varphi)}{h} \qquad \varphi \in \mathcal{D}.$$

To define the Lie bracket of $X, Y \in \mathcal{X}(\mathcal{D})$ we also proceed in local charts [35]. If $U \subset C^\infty(\mathbb{S})$ is open and $X, Y : U \to C^\infty(\mathbb{S})$ are smooth, we denote

$$D_X Y(\varphi) = \lim_{h\to 0} \frac{Y(\varphi + hX(\varphi)) - Y(\varphi)}{h} \qquad \varphi \in \mathcal{D}.$$

We are led to define the vector field

$$[X, Y] = D_X Y - D_Y X.$$

This definition is covariant and defines globally $\mathcal{L}_X Y = [X, Y]$.

Let $\mathcal{X}^R(\mathcal{D})$ be the space of all right-invariant smooth vector fields on $\mathcal{D}$. Note that $X \in \mathcal{X}^R(\mathcal{D})$ is determined by its value u at Id, $X(\eta) = R_\eta u$ for $\eta \in \mathcal{D}$, where R_η stands for the right translation. The bracket $[X, Y]$ of $X, Y \in \mathcal{X}^R(\mathcal{D})$ is a right-invariant vector field[7] and $[X, Y](Id) = [u, v]$, where $u = X(Id)$, $v = Y(Id)$ [35].

[6] The definition differs from the case of Banach spaces due to the fact that in general the space of linear maps of F_1 to F_2 will not form a Fréchet space. See [26] for a review of the intricacies of the Fréchet differential calculus.

[7] In view of the above discussion of $\exp_L$, every right-invariant vector field has a smooth flow on $\mathcal{D}$. The proof that the bracket preserves right-invariance is therefore standard.

2.2. Right-invariant metrics

$T_{Id}\mathcal{D} \equiv C^\infty(\mathbb{S})$ is not a Hilbert space. We define a weak right-invariant Riemannian metric on $\mathcal{D}$ as follows. We consider on $T_{Id}\mathcal{D} \equiv C^\infty(\mathbb{S})$ a nondegenerate continuous inner product $\langle\cdot,\cdot,\rangle$. That is, $u \mapsto \langle u, u\rangle$ is a continuous (hence smooth) map on $C^\infty(\mathbb{S})$ and the relation $\langle u, v\rangle = 0$ for all $v \in C^\infty(\mathbb{S})$ forces $u = 0$; a typical example would be the $H^s(\mathbb{S})$-inner product with $s \geqslant 0$. To define a smooth right-invariant Riemannian metric on $\mathcal{D}$, we extend this inner product to each tangent space $T_\eta\mathcal{D}$ by right translation, i.e.

$$\langle V, W\rangle(\eta) := \langle V \circ \eta^{-1}, W \circ \eta^{-1}\rangle \qquad \text{for} \quad V, W \in T_\eta\mathcal{D}. \tag{2.2}$$

Each open set of the topology induced by this inner product is open in the Fréchet space $C^\infty(\mathbb{S})$ but the converse is not true—we defined a weak topology on $C^\infty(\mathbb{S})$.

2.3. Covariant derivative

In order to define parallel translation along a curve on $\mathcal{D}$ and to derive the geodesic equation of the metric defined by (2.2), it is necessary to show the existence of a covariant derivative ∇ which preserves the inner product (2.2). Let us point out that, given a smooth Riemannian metric on $\mathcal{D}$, the existence of a metric covariant derivative is not ensured on general grounds as we deal with a Fréchet manifold. For the group of volume-preserving diffeomorphisms, the existence of the metric covariant derivative has been established in [22]. We shall see that a development related to the ideas considered in [1, 22] yields an existence result for the covariant derivative in the case of a right-invariant metric on $\mathcal{D}$. As the existence of such a covariant derivative is assumed in the literature [3], it is of interest to provide a rigorous proof for it.

Recall that a covariant derivative is defined as a $\mathbb{R}$-bilinear operator $\nabla : \mathcal{X}(\mathcal{D}) \times \mathcal{X}(\mathcal{D}) \to \mathcal{X}(\mathcal{D})$ with the following properties:

(i) $X(\eta) = 0$ implies $\nabla_X Y(\eta) = 0$ (punctual dependence in X),
(ii) $\nabla_X Y - \nabla_Y X = [X, Y]$ for $X, Y \in \mathcal{X}(\mathcal{D})$ (torsion free),
(iii) $\nabla_X(fY) = (\mathcal{L}_X f)Y + f\nabla_X Y$ for $f \in \mathcal{F}(\mathcal{D})$ and $X, Y \in \mathcal{X}(\mathcal{D})$, and for all $X, Y, Z \in \mathcal{X}(\mathcal{D})$,
(iv) $\mathcal{L}_X\langle Y, Z\rangle = \langle\nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$ (compatibility with the metric).

Observe that (i) and the $\mathbb{R}$-linearity in X force $\nabla_X Y$ to be $\mathcal{F}(\mathcal{D})$-linear in X. In finite dimensions, punctual dependence on X and $\mathcal{F}(\mathcal{D})$-linearity in X are equivalent but this cannot be ensured in infinite dimensions, [32] pp 202–3. Since $\mathcal{D}$ is a Fréchet manifold with a weak Riemannian metric, in general the existence of a covariant derivative is not ensured [3, 32]. A sufficient condition for the existence of a covariant derivative is given by

Theorem 1. *Assume that there exists a bilinear operator $B : C^\infty(\mathbb{S}) \times C^\infty(\mathbb{S}) \to C^\infty(\mathbb{S})$ such that*[8]

$$\langle B(u, v), w\rangle = \langle u, [v, w]\rangle \qquad u, v, w \in C^\infty(\mathbb{S}). \tag{2.3}$$

Then there exists a unique Riemannian connection ∇ on $\mathcal{D}$ associated with the right-invariant metric $\langle\cdot,\cdot\rangle$, given by

$$(\nabla_X Y)_\eta = \left[X, Y - Y^R_\eta\right]_\eta + \tfrac{1}{2}\left(\left[X^R_\eta, Y^R_\eta\right]_\eta - B\left(X^R_\eta, Y^R_\eta\right)_\eta - B\left(Y^R_\eta, X^R_\eta\right)_\eta\right)$$

[8] The operator B was introduced by Arnold [1] in the Lagrangian formulation of Euler's equation of motion of a perfect fluid in a bounded domain $\Omega \subset \mathbb{R}^3$. For Hilbert manifolds the existence of B is guaranteed by the Riesz representation theorem [32].

where for $X \in \mathcal{X}(\mathcal{D})$, we denote by X^R_η the right-invariant vector field whose value at η is X_η and we extend B to a bilinear map on the family $\mathcal{X}^R(\mathcal{D})$ of right-invariant vector fields, $B : \mathcal{X}^R(\mathcal{D}) \times \mathcal{X}^R(\mathcal{D}) \to \mathcal{X}^R(\mathcal{D})$ by $B(Z, W)_\eta = R_\eta B(Z_{Id}, W_{Id})$ for $\eta \in \mathcal{D}$ and $Z, W \in \mathcal{X}^R(\mathcal{D})$.

In the proof of theorem 1 we will use.

Lemma 1. *Consider on $\mathcal{D}$ a smooth right-invariant metric induced by an inner product $\langle\cdot,\cdot\rangle$. If $X, Y, Z \in \mathcal{X}(\mathcal{D})$ and $Y_\eta = 0$ at some $\eta \in \mathcal{D}$, then*

$$\mathcal{L}_X\langle Y, Z\rangle_\eta = \langle [X, Y], Z\rangle_\eta.$$

Proof. Write the relation to be proved as

$$(\mathcal{L}_X\langle R_{h^{-1}} Y_h, R_{h^{-1}} Z_h\rangle_e)(\eta) = \langle R_{\eta^{-1}}[X, Y]_\eta, R_{\eta^{-1}} Z_\eta\rangle_e$$

where R_φ stands for right translation and $e = Id$. Being in the Lie algebra of $\mathcal{D}$, we may specify

$$(\mathcal{L}_X\langle R_{h^{-1}} Y_h, R_{h^{-1}} Z_h\rangle_e)(\eta) = \langle D_X(R_{h^{-1}} Y(h))(\eta), R_{\eta^{-1}} Z_\eta\rangle_e$$

so that is suffices to show that $D_X(R_{h^{-1}} Y(h))(\eta) = R_{\eta^{-1}}[X, Y]_\eta$. This last relation is true in a Hilbert space H as we can derive $R_{\eta^{-1}}$ which belongs to the Hilbert space $\mathcal{L}(H, H)$ of continuous linear operators from H to H (note that $\mathcal{L}(C^\infty(\mathbb{S}), C^\infty(\mathbb{S}))$ is not a Fréchet space [26]). Therefore the last equality holds in each $H^k(\mathbb{S})$, $k \geqslant 2$, and we infer the result from this if we take into account the definition of convergence on $C^\infty(\mathbb{S})$. □

Proof of theorem 1. As the proof is rather elaborate, we proceed in several steps. We first show that uniqueness is ensured. Assuming the existence of ∇, we derive its expression on right-invariant vector fields and show that this completely determines ∇. Our last task will be to show that the obtained explicit formula for ∇ satisfies properties (i)–(iv).

Step 1. We show the uniqueness of ∇ and, assuming existence, we derive its expression on right-invariant vector fields.

Let us write (iv) for a cyclic permutation of $X, Y, Z \in \mathcal{X}(\mathcal{D})$,

$$\mathcal{L}_X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle \nabla_X Z, Y\rangle$$
$$\mathcal{L}_Y\langle X, Z\rangle = \langle \nabla_Y Z, X\rangle + \langle \nabla_Y X, Z\rangle$$
$$\mathcal{L}_Z\langle X, Y\rangle = \langle \nabla_Z X, Y\rangle + \langle \nabla_Z Y, X\rangle.$$

Adding the first two relations and subtracting the third, the following identity can be derived:

$$2\langle \nabla_X Y, Z\rangle = \langle [X, Y], Z\rangle - \langle [Y, Z], X\rangle + \langle [Z, X], Y\rangle - \mathcal{L}_X\langle Y, Z\rangle - \mathcal{L}_Y\langle X, Z\rangle + \mathcal{L}_Z\langle X, Y\rangle \tag{2.4}$$

if we take (ii) into account. Since the inner product $\langle\cdot,\cdot\rangle$ is nondegenerate, the previous formula shows the uniqueness of ∇.

Let $\mathcal{X}^R(\mathcal{D})$ be the space of all right-invariant smooth vector fields on $\mathcal{D}$. Due to the right-invariance of the metric, $\langle Y, Z\rangle$ is constant for $Y, Z \in \mathcal{X}^R(\mathcal{D})$ so that $\mathcal{L}_X\langle Y, Z\rangle = 0$ for all $X \in \mathcal{X}(\mathcal{D})$. Therefore, (2.4) reduces to

$$2\langle \nabla_X Y, Z\rangle = \langle [X, Y], Z\rangle - \langle [Y, Z], X\rangle + \langle [Z, X], Y\rangle \qquad X, Y, Z \in \mathcal{X}^R(\mathcal{D}).$$

We evaluate this relation at $e = Id$ to obtain by means of (2.3)

$$\begin{aligned} 2\langle R_{\eta^{-1}}(\nabla_X Y)_\eta, Z_e\rangle_e &= \langle [X, Y]_e, Z_e\rangle_e - \langle [Y, Z]_e, X_e\rangle_e + \langle [Z, X]_e, Y_e\rangle_e \\ &= \langle [X, Y]_e, Z_e\rangle_e - \langle B(X_e, Y_e), Z_e\rangle_e - \langle B(Y_e, X_e), Z_e\rangle_e \end{aligned}$$

as $R_{\eta^{-1}} X_\eta = X_e$ by right-invariance and since the Lie bracket of two right-invariant vector fields is a right-invariant vector field. We get

$$(\nabla_X Y)_\eta = \tfrac{1}{2}([X, Y]_\eta - R_\eta B(X_e, Y_e) - R_\eta B(Y_e, X_e)) \qquad \eta \in \mathcal{D} \tag{2.5}$$

which is the expression of ∇ on right-invariant vector fields.

Step 2. Assuming the existence of ∇ we derive an explicit formula for it.

If $X \in \mathcal{X}(\mathcal{D})$, we denote by X^R_η the right-invariant vector field on $\mathcal{D}$ whose value at $\eta \in \mathcal{D}$ is X_η. If ∇ exists, then

$$(\nabla_X Y)_\eta = \left[X, Y - Y^R_\eta\right]_\eta + \left(\nabla_{X^R_\eta} Y^R_\eta\right)_\eta \qquad \eta \in \mathcal{D}. \tag{2.6}$$

Indeed, by (ii) ∇ must be torsion free so that

$$\left(\nabla_X \left(Y - Y^R_\eta\right)\right)(\eta) - \left(\nabla_{\left(Y - Y^R_\eta\right)} X\right)(\eta) = \left[X, Y - Y^R_\eta\right]_\eta$$

and (i) yields $\left(\nabla_X\left(Y - Y^R_\eta\right)\right)(\eta) = \left[X, Y - Y^R_\eta\right]_\eta$. Combining this with (i) we obtain

$$(\nabla_X Y)_\eta = \left[X, Y - Y^R_\eta\right]_\eta + \left(\nabla_X Y^R_\eta\right)_\eta = \left[X, Y - Y^R_\eta\right]_\eta + \left(\nabla_{X^R_\eta} Y^R_\eta\right)_\eta \tag{2.7}$$

which is the only possible formula for ∇.

Step 3. We define ∇ by (2.7) and check that it satisfies all required properties (i)–(iv).

It is useful to write (2.7) in the more detailed form

$$(\nabla_X Y)_\eta = \left[X, Y - Y^R_\eta\right]_\eta + \frac{1}{2}\left(\left[X^R_\eta, Y^R_\eta\right]_\eta - B\left(X^R_\eta, Y^R_\eta\right)_\eta - B\left(Y^R_\eta, X^R_\eta\right)_\eta\right)$$

where we extended B to a bilinear map $B : \mathcal{X}^R(\mathcal{D}) \times \mathcal{X}^R(\mathcal{D}) \to \mathcal{X}^R(\mathcal{D})$ by $B(Z, Z')_\eta = R_\eta B(Z_e, Z'_e)$ for $\eta \in \mathcal{D}$ and $Z, Z' \in \mathcal{X}^R(\mathcal{D})$. Since the vector field $\left(Y - Y^R_\eta\right)$ is zero at η, we have $\left[X, Y - Y^R_\eta\right]_\eta = \left[X^R_\eta, Y - Y^R_\eta\right]_\eta$ as one can see by going to local charts. We therefore have a second equivalent explicit form of (2.7),

$$(\nabla_X Y)_\eta = \left[X^R_\eta, Y - Y^R_\eta\right]_\eta + \frac{1}{2}\left(\left[X^R_\eta, Y^R_\eta\right]_\eta - B\left(X^R_\eta, Y^R_\eta\right)_\eta - B\left(Y^R_\eta, X^R_\eta\right)_\eta\right).$$

Clearly, ∇ is $\mathbb{R}$-bilinear. The above explicit form of (2.7) shows that $(\nabla_X Y)_\eta$ depends only on the value X_η of X at η. Property (iii) can be easily checked as the expression $\left(\left[X^R_\eta, Y^R_\eta\right]_\eta - B\left(X^R_\eta, Y^R_\eta\right)_\eta - B\left(Y^R_\eta, X^R_\eta\right)_\eta\right)$ is tensorial.

To verify that ∇ is torsion free, note that the above two explicit forms of (2.7) yield $(\nabla_X Y)_\eta = \left(\nabla_{X^R_\eta} Y\right)_\eta$ so that, by these formulae

$$\begin{aligned}(\nabla_X Y - \nabla_Y X)_\eta &= \left(\nabla_{X^R_\eta} Y - \nabla_Y X\right)_\eta \\ &= \left[X^R_\eta, Y - Y^R_\eta\right]_\eta - \left[Y, X - X^R_\eta\right]_\eta + \left[X^R_\eta, Y^R_\eta\right]_\eta\end{aligned}$$

which cancels to $[X, Y]_\eta$.

To complete the proof, we have to check that ∇ defined by (2.7) is compatible with the metric. To prove (iv) at a given $\eta \in \mathcal{D}$ we have to show that

$$\mathcal{L}_X \langle Y, Z\rangle_\eta = \left\langle\left[X^R_\eta, Y - Y^R_\eta\right], Z\right\rangle_\eta + \left\langle\left[X^R_\eta, Z - Z^R_\eta\right], Y\right\rangle_\eta$$

as the remaining parts cancel. Due to bilinearity, it will be enough to verify the above equality for the triples $\left(X, Y - Y^R_\eta, Z\right)$, $\left(X, Y^R_\eta, Z - Z^R_\eta\right)$ and $\left(X, Y^R_\eta, Z^R_\eta\right)$. The first two triples satisfy the equality in view of lemma 1 while for the third triple the verification is obvious as both sides are zero.

Therefore we proved that there exists a unique Riemannian connection ∇ on $\mathcal{D}$ associated with the right-invariant metric $\langle\cdot, \cdot\rangle$. From its explicit form we see that ∇ maps right-invariant vector fields onto right-invariant vector fields. $\square$

2.4. Derivative along a curve and parallelism

Let us now construct a *derivation along curves.* Let $J \subset \mathbb{R}$ be an open interval and consider a C^1-curve $\alpha : J \to \mathcal{D}$. By a lift γ of α we mean a C^1-curve $\gamma : J \to T\mathcal{D}$ lying above α. If $\mathrm{Lift}(\alpha)$ is the set of lifts of α, we define the derivation $D_{\alpha_t} : \mathrm{Lift}(\alpha) \to \mathrm{Lift}(\alpha)$ along α in local coordinates by

$$D_{\alpha_t}\gamma = \gamma_t - Q(\alpha_t \circ \alpha^{-1}, \gamma \circ \alpha^{-1}) \circ \alpha \qquad \gamma \in \mathrm{Lift}(\alpha) \tag{2.8}$$

where the bilinear operator $Q : C^\infty(\mathbb{S}) \times C^\infty(\mathbb{S}) \to C^\infty(\mathbb{S})$ is defined by

$$Q(u, v) = \tfrac{1}{2}(u_x v + u v_x + B(u, v) + B(v, u)) \qquad u, v \in C^\infty(\mathbb{S}).$$

If α is induced by a vector field, we recover the expression of the covariant derivative. Indeed, let $X, Y \in \mathcal{X}(\mathcal{D})$ be such that $\gamma(t) = Y(\alpha(t))$ on J and $\alpha_t(t_0) = X(\alpha_0)$, $\alpha_0 = \alpha(t_0)$ for some $t_0 \in J$. If X_0^R, Y_0^R are the right-invariant vector fields on $\mathcal{D}$ whose values at $\alpha_0 \in \mathcal{D}$ are $X(\alpha_0)$, respectively $Y(\alpha_0)$, we have, cf step 3 in the proof of theorem 1, that

$$(\nabla_X Y)(\alpha_0) = \left[X_0^R, Y\right](\alpha_0) - \tfrac{1}{2}\left(\left[X_0^R, Y_0^R\right](\alpha_0) + B\left(X_0^R, Y_0^R\right)(\alpha_0) + B\left(Y_0^R, X_0^R\right)(\alpha_0)\right).$$

According to section 2.1, in local coordinates,

$$\left[X_0^R, Y_0^R\right](\alpha_0) = \gamma(t_0) \cdot \left(\left[X(\alpha_0) \circ \alpha_0^{-1}\right]_x \circ \alpha_0\right) - X(\alpha_0) \cdot \left(\left[\gamma(t_0) \circ \alpha_0^{-1}\right]_x \circ \alpha_0\right).$$

On the other hand, writing out the definition explicitly, we see that

$$\left[X_0^R, Y\right](\alpha_0) = \gamma_t(t_0) - \gamma(t_0) \cdot \left(\left[X(\alpha_0) \circ \alpha_0^{-1}\right]_x \circ \alpha_0\right)$$

thus $D_{\alpha_t}\gamma(t_0) = (\nabla_X Y)(\alpha(t_0))$. Let us now prove

Lemma 2. *Let $J \subset \mathbb{R}$ be an open interval and consider a C^1-curve $\alpha : J \to \mathcal{D}$. If $\gamma_1, \gamma_2 \in \mathrm{Lift}(\alpha)$, then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \gamma_1, \gamma_2 \rangle = \left\langle D_{\alpha_t}\gamma_1, \gamma_2 \right\rangle + \left\langle \gamma_1, D_{\alpha_t}\gamma_2 \right\rangle \qquad t \in J. \tag{2.9}$$

Proof. The method is quite similar to that we used in the case of vector fields. Let us fix $t_0 \in J$. First we establish, in the same way as in lemma 1, that

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \gamma_1, \gamma_2 \rangle\Big|_{t=t_0} = \left\langle \left(\frac{\mathrm{d}}{\mathrm{d}t}\gamma_1\right)(t_0), \gamma_2(t_0) \right\rangle \qquad \text{if} \quad \gamma_1(t_0) = 0. \tag{2.10}$$

Then we prove that (2.9) is satisfied at $t = t_0$ by the three couples

$$\left(\gamma_1 - \gamma_1^R, \gamma_2\right) \qquad \left(\gamma_1^R, \gamma_2 - \gamma_2^R\right) \qquad \left(\gamma_1^R, \gamma_2^R\right)$$

where $\gamma_i^R(t) = R_{\alpha(t)} R_{\alpha(t_0)^{-1}} \gamma_i(t_0)$, $t \in J$, $i = 1, 2$.

Indeed, defining the right-invariant vector fields Y_i^R whose values at $\alpha(t_0)$ are $\gamma_i(t_0)$, observe that $\gamma_i^R(t) = Y_i^R(\alpha(t))$, $t \in J$, $i = 1, 2$. That (2.9) is true for the first two couples is a direct consequence of (2.10). On the other hand, since γ_i^R derive from the vector fields Y_i^R, by the compatibility of the covariant derivative with the metric we have

$$\begin{aligned}\mathcal{L}_{X^R}\left\langle Y_1^R, Y_2^R\right\rangle_{\alpha(t)} &= \left\langle \nabla_{X^R} Y_1^R, Y_2^R\right\rangle_{\alpha(t)} + \left\langle Y_1^R, \nabla_{X^R} Y_2^R\right\rangle_{\alpha(t)} \\ &= \left\langle D_{\alpha_t}\gamma_1^R(t), \gamma_2^R(t)\right\rangle + \left\langle \gamma_1^R(t), D_{\alpha_t}\gamma_2^R(t)\right\rangle\end{aligned}$$

where X^R is the right-invariant vector field on $\mathcal{D}$ whose value at $\alpha(t_0)$ is $\alpha_t(t_0)$. But

$$\mathcal{L}_{X^R}\left\langle Y_1^R, Y_2^R\right\rangle_{\alpha(t_0)} = \frac{\mathrm{d}}{\mathrm{d}t}\left\langle \gamma_1^R, \gamma_2^R\right\rangle\Big|_{t=t_0}$$

as one can check using the fact that X^R has a flow, see the discussion of the Lie group exponential map on $\mathcal{D}$. Therefore the third couple satisfies (2.9) at $t = t_0$ too. Adding up these three relations, we obtain (2.9) at $t = t_0$. Due to the arbitrariness of $t_0 \in \mathcal{D}$, the proof is complete. □

If $\varphi : J \to \mathcal{D}$ is a C^2-curve, we say that a lift $\gamma : J \to T\mathcal{D}$ is *φ-parallel* if $D_{\varphi_t}\gamma \equiv 0$ on J. In local coordinates, denoting

$$\varphi_t \circ \varphi^{-1} = u \qquad \gamma \circ \varphi^{-1} = v$$

this is equivalent to requiring that $v \in C^1(J; C^\infty(\mathbb{S}))$ is a solution of the equation

$$v_t = \tfrac{1}{2}(vu_x - v_x u + B(u, v) + B(v, u)). \tag{2.11}$$

A C^2-curve $\varphi : J \to \mathcal{D}$ is called a *geodesic* if $D_{\varphi_t}\varphi_t \equiv 0$ on J. With $u = \varphi_t \circ \varphi^{-1} \in T_{Id}\mathcal{D} \equiv C^\infty(\mathbb{S})$ we can write the geodesic equation as

$$u_t = B(u, u) \qquad t \in J. \tag{2.12}$$

Both (2.11) and (2.12) are differential equations in the Fréchet space $C^\infty(\mathbb{S})$. The classical local existence theorem for differential equations with smooth right-hand side does not hold in $C^\infty(\mathbb{S})$ [26]. We adopt the following approach. We complete $C^\infty(\mathbb{S})$ under the $H^k(\mathbb{S})$-norm ($k \geqslant 2$), deal with the resulting Hilbert manifold $\mathcal{D}^k$, and then show that the solutions of the equation under study actually are C^∞ if the data are smooth. More precisely, for $k \geqslant 2$, let

$$\mathcal{D}^k = \{\eta \in H^k(\mathbb{S}), \eta \text{ is bijective, orientation preserving and } \eta^{-1} \in H^k(\mathbb{S})\}.$$

The $\mathcal{D}^k$, $k \geqslant 2$, is only a topological group and is not a Lie group as the composition map

$$\mathcal{D}^k \times \mathcal{D}^k \to \mathcal{D}^k \qquad (f, g) \mapsto f \circ g$$

and the inverse map

$$\mathcal{D}^k \to \mathcal{D}^k \qquad f \mapsto f^{-1}$$

are merely continuous, not C^∞. For $\varphi \in \mathcal{D}^k$, right composition

$$R_\varphi : \mathcal{D}^k \to \mathcal{D}^k \qquad R_\varphi(\eta) = \eta \circ \varphi \qquad \eta \in \mathcal{D}^k$$

is a C^∞ map but left composition

$$L_\varphi : \mathcal{D}^k \to \mathcal{D}^k \qquad L_\varphi(\eta) = \varphi \circ \eta \qquad \eta \in \mathcal{D}^k$$

is continuous without being locally Lipschitz. However, the composition regarded as a map $\mathcal{D}^{k+n} \times \mathcal{D}^k \to \mathcal{D}^k$ and the group inverse regarded as a map $\mathcal{D}^{k+n} \to \mathcal{D}^k$ are both of class C^n. $\mathcal{D}^k$, $k \geqslant 2$, is a Hilbert manifold modelled on $T_{Id}\mathcal{D}^k \equiv H^k(\mathbb{S})$; see [23] for a detailed treatment of these matters. In our approach, the study of the structure of all the $\mathcal{D}^k$, $k \geqslant 2$, with respect to a given right-invariant metric will enable us to obtain results for $\mathcal{D}$ as the geodesic flow on $\mathcal{D}^k$ preserves $\mathcal{D}$.

3. The L^2 right-invariant metric

Since $T_{Id}\mathcal{D}$ is a smooth function space, the most natural inner product to start with would be the L^2 inner product

$$\langle u, v\rangle_{L^2} = \int_{\mathbb{S}} u(x)v(x)\,\mathrm{d}x \quad \text{on} \quad T_{Id}\mathcal{D} \equiv C^\infty(\mathbb{S}).$$

In the case of the smooth right-invariant metric obtained by right translation by means of (2.2), it is easy to check that $B(u, v) = -2u'v - uv'$ for $u, v \in C^\infty(\mathbb{S})$. The geodesic equation for the L^2 right-invariant metric is

$$u_t + 3uu_x = 0 \tag{3.1}$$

where $t \mapsto \varphi(t)$ is the geodesic curve starting at time $t = 0$ at the identity Id in the direction $u_0 \in T_{Id}\mathcal{D}, u = \varphi_t \in T_{\varphi(t)}\mathcal{D}$. Note that (3.1) is a differential equation in $C^\infty(\mathbb{S})$. Equation (3.1) is part of the system

$$\begin{cases} \varphi_t = u(t, \varphi) \\ u_t + 3uu_x = 0 \end{cases} \tag{3.2}$$

with initial data $\varphi(0) = Id, u_0 \in C^\infty(\mathbb{S})$.

3.1. Burgers equation

Equation (3.1) is the well-known inviscid Burgers equation [7]. Though rather simple, it is a successful mathematical model of gas dynamics [4]. This partial differential equation was investigated in great detail. If $u_0 \in H^k(\mathbb{S})$ with $k \geqslant 2$, then equation (3.1) with initial data $u(0, \cdot) = u_0$ has a unique solution $u \in C([0, T); H^k(\mathbb{S})) \cap C^1([0, T); H^{k-1}(\mathbb{S}))$ for some maximal time $T > 0$ [31]. Moreover, on $[0, T), u(t)$ depends continuously on the initial data in the $H^k(\mathbb{S})$-norm, while Hölder continuity with any prescribed exponent generally does not hold, see [30]. Equation (3.1) can be analysed by the method of characteristics. If $u_0 \in H^k(\mathbb{S}), k \geqslant 2$, then the solution $u \in C([0, T); H^k(\mathbb{S}))$ satisfies

$$u(t, x + 3t\, u_0(x)) = u_0(x) \qquad t \in [0, T) \quad x \in \mathbb{S}. \tag{3.3}$$

Using this, one can see[9] that the maximal existence time is precisely

$$T = \min_{\{x \in \mathbb{S}: u_0'(x) < 0\}} \left\{ \frac{1}{-3u_0'(x)} \right\} > 0. \tag{3.4}$$

Since u_0 is periodic, we deduce that all solutions but the constant functions have a finite life-span. The development of singularities is also well understood: if $u_0 \in H^k(\mathbb{S}), k \geqslant 2$, is not constant, then

$$\max_{x \in \mathbb{S}} |u(t, x)| = \max_{x \in \mathbb{S}} |u_0(x)| \qquad t \in [0, T)$$

while

$$\min_{x \in \mathbb{S}} u_x(t, x) \to -\infty \quad \text{as} \quad t \uparrow T < \infty.$$

Note that on $[0, T)$ we have $u(t, \cdot) \in H^2(\mathbb{S}) \subset C^1(\mathbb{S})$. Relation (3.3) is useful in determining the blow-up rate

$$\lim_{t \uparrow T} \left((T - t) \min_{x \in \mathbb{S}} \{u_x(t, x)\} \right) = -\tfrac{1}{3}.$$

3.2. Existence of geodesics

It is quite natural to view (3.1) as the geodesic equation for the right-invariant L^2-metric on $\mathcal{D}^k$ with $k \geqslant 2$. However, this needs further justification since, in contrast to the case of $\mathcal{D}$, we cannot start from the notion of covariant derivative to define the geodesics. Note that the alleged covariant derivative given by theorem 1 is not well defined on $\mathcal{D}^k$ due to loss of smoothness. We would also like to point out that if $X \in \mathcal{X}(\mathcal{D}^k), k \geqslant 2$, then the map $\eta \mapsto X(\eta) \circ \eta^{-1}$ is only continuous on $\mathcal{D}^k$ so that the L^2 right-invariant metric on $\mathcal{D}^k$ *is not smooth* whereas the L^2 right-invariant metric on $\mathcal{D}$ is smooth. To fully justify why we are entitled to call (3.1) the geodesic equation on $\mathcal{D}^k$, we will show that it arises from the necessary condition for a curve on $\mathcal{D}^k$ to be locally length minimizing.

[9] For details we refer to [28].

For each C^1-curve $\gamma : [a, b] \to \mathcal{D}$ we define its length by

$$l(\gamma) = \int_a^b \|\gamma_t(t)\|_{\gamma(t)}\,\mathrm{d}t = \int_a^b \langle \gamma_t \circ \gamma^{-1}, \gamma_t \circ \gamma^{-1}\rangle^{\frac{1}{2}}\,\mathrm{d}t.$$

We can extend the length to piecewise C^1-paths on $\mathcal{D}$ by taking the sum of the lengths of the C^1-components of the curve. Since $\mathcal{D}$ is connected, cf section 2.1, any two points on $\mathcal{D}$ can be joined by a piecewise C^1-path. We say that a C^1-path $\gamma : [a, b] \to \mathcal{D}$ has a regular parametrization if $\gamma_t \neq 0$ at every $t \in [a, b]$. Any such curve can be reparametrized by arc length [32], i.e. there is a new parametrization $\varphi : [0, c] \to \mathcal{D}$ of the path such that $\|\varphi_t\|_{\varphi(t)} = 1$ on $[0, c]$. The action along a path $\gamma : [a, b] \to \mathcal{D}$ is the quantity $\mathfrak{a}(\gamma) = \frac{1}{2}\int_a^b \|\gamma_t\|^2_{\varphi(t)}\,\mathrm{d}t$. Unlike the length $l(\gamma)$, the action $\mathfrak{a}(\gamma)$ depends on the parametrization. If the curve is parametrized by arc length, we have $l(\gamma) = 2\mathfrak{a}(\gamma)$. This allows us to pass freely from one notion to the other. Let us now find a necessary condition for a regularly parametrized path to be the shortest path on $\mathcal{D}$ between its endpoints. In view of the previous comments, we can assume the path to be parametrized by arc length, $\gamma : [0, c] \to \mathcal{D}$ and γ is a critical point in the space of paths for the action functional, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\mathfrak{a}(\gamma + \epsilon\eta)\Big|_{\epsilon=0} = 0$$

for every path $\eta : [0, c] \to \mathcal{D}$ with endpoints at zero and such that $\gamma + \epsilon\eta$ is a small variation of γ on $\mathcal{D}$. But

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\epsilon}\mathfrak{a}(\gamma + \epsilon\eta)\Big|_{\epsilon=0} &= \frac{\mathrm{d}}{\mathrm{d}\epsilon}\frac{1}{2}\int_0^c\int_{\mathbb{S}}[(\gamma_t + \epsilon\eta_t) \circ (\gamma + \epsilon\eta)^{-1}]^2\,\mathrm{d}x\,\mathrm{d}t\Big|_{\epsilon=0}\\ &= \int_0^c\int_{\mathbb{S}}(\gamma_t \circ \gamma^{-1})\left\{\frac{\mathrm{d}}{\mathrm{d}\epsilon}[(\gamma_t + \epsilon\eta_t) \circ (\gamma + \epsilon\eta)^{-1}]\Big|_{\epsilon=0}\right\}\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Note that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\epsilon}[(\gamma_t + \epsilon\eta_t) \circ (\gamma + \epsilon\eta)^{-1}]\Big|_{\epsilon=0} &= \eta_t \circ \gamma^{-1} + (\gamma_{tx} \circ \gamma^{-1}) \cdot \frac{\mathrm{d}}{\mathrm{d}\epsilon}(\gamma + \epsilon\eta)^{-1}\Big|_{\epsilon=0}\\ &= \eta_t \circ \gamma^{-1} - (\gamma_{tx} \circ \gamma^{-1})\frac{\eta \circ \gamma^{-1}}{\gamma_x \circ \gamma^{-1}}\end{aligned}$$

as differentiation with respect to ϵ in the relation $(\gamma + \epsilon\eta) \circ (\gamma + \epsilon\eta)^{-1} = Id$ leads to

$$\gamma_x \circ (\gamma + \epsilon\eta)^{-1} \cdot \frac{\mathrm{d}}{\mathrm{d}\epsilon}(\gamma + \epsilon\eta)^{-1} + \epsilon\eta_x \circ (\gamma + \epsilon\eta)^{-1} \cdot \frac{\mathrm{d}}{\mathrm{d}\epsilon}(\gamma + \epsilon\eta)^{-1} + \eta \circ (\gamma + \epsilon\eta)^{-1} = 0$$

and therefore

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}(\gamma + \epsilon\eta)^{-1}\Big|_{\epsilon=0} = -\frac{\eta \circ \gamma^{-1}}{\gamma_x \circ \gamma^{-1}}.$$

We infer that

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\mathfrak{a}(\gamma + \epsilon\eta)\Big|_{\epsilon=0} = \int_0^c\int_{\mathbb{S}}(\gamma_t \circ \gamma^{-1})[\eta_t \circ \gamma^{-1} - (\eta \circ \gamma^{-1})\partial_x(\gamma_t \circ \gamma^{-1})]\,\mathrm{d}x\,\mathrm{d}t.$$

Denoting $\gamma_t \circ \gamma^{-1} = u$, we find

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\epsilon}\mathfrak{a}(\gamma + \epsilon\eta)\Big|_{\epsilon=0} &= \int_0^c\int_{\mathbb{S}} u[\eta_t \circ \gamma^{-1} - u_x\eta \circ \gamma^{-1}]\,\mathrm{d}x\,\mathrm{d}t\\ &= \int_0^c\int_{\mathbb{S}} u[\partial_t(\eta \circ \gamma^{-1}) + u\partial_x(\eta \circ \gamma^{-1}) - u_x(\eta \circ \gamma^{-1})]\,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

since

$$\partial_t(\eta \circ \gamma^{-1}) = \eta_t \circ \gamma^{-1} + \eta_x \circ \gamma^{-1} \cdot \partial_t(\gamma^{-1}) = \eta_t \circ \gamma^{-1} - \eta_x \circ \gamma^{-1} \cdot \frac{\gamma_t \circ \gamma^{-1}}{\gamma_x \circ \gamma^{-1}}.$$

Indeed, differentiating the relation $\gamma \circ \gamma^{-1} = Id$ with respect to time, we get $\gamma_t \circ \gamma^{-1} + \gamma_x \circ \gamma^{-1} \cdot \partial_t(\gamma^{-1}) = 0$ which gives the desired expression for $\partial_t(\gamma^{-1})$. Integrating by parts with respect to t and x in the above formula for the derivative of the action functional, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\mathfrak{a}(\gamma + \epsilon\eta)\Big|_{\epsilon=0} = -\int_0^c \int_{\mathbb{S}} (\eta \circ \gamma^{-1})[u_t + 3uu_x]\,\mathrm{d}x\,\mathrm{d}t.$$

This calculation can be performed on $\mathcal{D}$ as well as on $\mathcal{D}^k, k \geqslant 2$, and yields the Euler–Lagrange equation $u_t + 3uu_x = 0$ where $u = \gamma_t \circ \gamma^{-1}$ and $t \mapsto \gamma(t) \in \mathcal{D}$ is the curve (parametrized by arc length) yielding the critical point of the length functional to be minimized. The variational formulation gives a meaning to the geodesic equation on $\mathcal{D}^k, k \geqslant 2$. To proceed, we shall need

Lemma 3. [6] *Let $F \in C([0,T); H^k(\mathbb{S}))$ with $k \geqslant 2$. Then the differential equation*

$$\begin{cases} \varphi_t = F(t, \varphi) \\ \varphi(0) = Id \end{cases}$$

has a unique solution $\varphi \in C^1([0,T); \mathcal{D}^k)$.

The considerations in section 3.1 show that for any $u_0 \in H^k(\mathbb{S}), k \geqslant 2$, the system (3.2) defines a unique C^1-curve $t \mapsto \varphi(t) \in \mathcal{D}^k$ on a maximal time interval $[0,T)$ with $T > 0$ given by (3.4). Note that we obtained the geodesic equation (3.1)—a geodesic for the L^2 right-invariant metric being defined to be a C^1-curve satisfying (3.1). The discussion in section 2.4 would suggest defining geodesics as C^2-curves $t \mapsto \varphi(t) \in \mathcal{D}^k$ satisfying (3.1) whereas our approach yields only a C^1-dependence on time. It is not possible to require $\varphi \in C^2([0,T); H^k(\mathbb{S}))$ as this assumption would lead by (3.2) to $\varphi_{tt} = u_t \circ \varphi + u \circ \varphi \cdot u_x \circ \varphi = -2u \circ \varphi \cdot u_x \circ \varphi \in C([0,T); H^k(\mathbb{S}))$. Letting in this relation $t \downarrow 0$ we would obtain that $u_0 \cdot u_0' \in H^k(\mathbb{S})$ for all $u_0 \in H^k(\mathbb{S})$, a contradiction.

Inspecting the previous considerations it becomes clear that we proved

Proposition 1. *For every $u_0 \in T_{Id}\mathcal{D}^k \equiv H^k(\mathbb{S}), k \geqslant 2$, there exists a unique geodesic on $\mathcal{D}^k$ starting at Id in the direction of u_0. This geodesic is defined for some finite maximal time $T > 0$ unless u_0 is constant.*

From the detailed discussion of equation (3.1) we know that if $u_0 \in C^\infty(\mathbb{S})$, then the unique solution u of (3.1) with data u_0 belongs to $C^1([0,T); \mathcal{D})$ with T given by (3.4). By a recursive argument using (3.1) we deduce that $u \in C^\infty([0,T); \mathcal{D})$. Applying lemma 3 for all $k \geqslant 2$ we obtain again by a recursive argument that

Theorem 2. *For every $u_0 \in T_{Id}\mathcal{D} \equiv C^\infty(\mathbb{S})$, there exists a unique geodesic $\varphi \in C^\infty([0,T); \mathcal{D})$ starting at Id in the direction of u_0. The only geodesic that can be continued indefinitely in time is that in the constant direction.*

For every $u_0 \in H^k(\mathbb{S}), k \geqslant 2$, we defined a geodesic curve $\varphi \in C^1([0,T), \mathcal{D}^k)$ starting at Id. On the other hand, the method of characteristics also associates a C^1-curve $t \mapsto q(t)$ on $\mathcal{D}^k$ starting at Id by

$$q(t,x) = x + 3t\,u_0(x) \qquad t \in [0,T) \quad x \in \mathbb{S}.$$

Generally the two curves do not coincide (if u_0 is constant, we have the same curve). While $t \mapsto q(t)$ satisfies (3.3), note that

$$u(t, \varphi(t,x)) \cdot \varphi_x^2(t,x) = u_0(x) \qquad t \in [0,T) \quad x \in \mathbb{S} \tag{3.5}$$

as one can see differentiating both sides with respect to time[10]. Observe that the solution to (3.1) is given by $u = \varphi_t \circ \varphi^{-1} = u_0 \circ q^{-1}$ up to the maximal existence time given by (3.4): the geometric approach differs from the method of characteristics.

3.3. The exponential map

These results enable us to define the *Riemannian exponential map* $\mathfrak{exp}$ of the L^2 right-invariant metric. Let $\varphi(t; u_0)$ be the geodesic starting at Id in the direction u_0 on $\mathcal{D}^k$, $k \geqslant 2$, or on $\mathcal{D}$. For later use, let us first observe that, using (3.2), it is easy to obtain

$$\varphi(t; su_0) = \varphi(ts; u_0) \tag{3.6}$$

for $t, s \geqslant 0$ such that both geodesics are well defined. On the other hand, note that $\|u_0\|_{H^k} < \frac{1}{4}$ ensures that the maximal existence time of $\varphi(t; u_0)$ is strictly larger than 1. Indeed, by the inequality

$$\max_{x\in\mathbb{S}} |u_0'(x)|^2 \leqslant \int_{\mathbb{S}} \left((u_0')^2 + (u_0'')^2\right) \mathrm{d}x \leqslant \|u_0\|^2_{H^k}$$

we obtain that $\max_{x\in\mathbb{S}}\{-3u_0'(x)\} \leqslant \frac{3}{4}$ so that the assertion follows from relation (3.4). For $\|u_0\|_{H^k} < \frac{1}{4}$ we define the Riemannian exponential map $\mathfrak{exp}$ as the time one map of the geodesic flow, i.e. $\mathfrak{exp}(u_0) = \varphi(1; u_0)$.

For strong Riemannian manifolds, the Riemannian exponential map always defines charts [32]. This is not the case for the (weak) L^2 right-invariant metric.

Proposition 2. *The Riemannian exponential map of the L^2 right-invariant metric on $\mathcal{D}^k$, $k \geqslant 2$, is not a C^1 map from a neighbourhood of zero in $T_{Id}\mathcal{D}^k \equiv H^k(\mathbb{S})$ to $\mathcal{D}^k$.*

Proof. Assuming the contrary, we will reach a contradiction by showing that although the derivative of $\mathfrak{exp}$ is the identity at zero, it fails to be invertible at nearby points. This will prove the assertion, for if $\mathfrak{exp}$ were C^1, the inverse function theorem would prevent this degeneracy.

We assume that $\mathfrak{exp}$ is a C^1 map.

Let $t \mapsto tv$ be a curve in $T_{Id}\mathcal{D}^k$. For $t > 0$ small enough, we have by (3.6) that $\mathfrak{exp}(tv) = \varphi(1; tv) = \varphi(t; v)$ so that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{exp}(tv)\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\varphi(t; v)\Big|_{t=0} = v \qquad v \in T_{Id}\mathcal{D}^k.$$

This shows that $D\mathfrak{exp}(0)$ is the identity.

We shall now compute the derivative of $\mathfrak{exp}$ at a point $v \in T_{Id}\mathcal{D}^k$ near Id by considering an infinitesimal change w of v. Denoting

$$\psi(t) = tD\mathfrak{exp}(tv) \cdot w \in H^k(\mathbb{S}) \qquad t \in [0, 1]$$

we will show that for $t \in [0, 1]$ we have

$$\psi(t, x) = \int_0^t \frac{w(x)}{\varphi_x^2(s, x)}\,\mathrm{d}s - \int_0^t \frac{2v(x)}{\varphi_x^3(s, x)}\psi_x(s, x)\,\mathrm{d}s \qquad x \in \mathbb{S}. \tag{3.7}$$

From its definition we know that $\psi(t, x)$ depends continuously on time while the Sobolev imbedding $H^{k-1}(\mathbb{S}) \subset C(\mathbb{S})$ shows the C^1-dependence on the spatial variable. Differentiating the above equation with respect to time we obtain the linear partial differential equation

$$\psi_t(t, x) = \frac{w(x)}{\varphi_x^2(t, x)} - \frac{2v(x)}{\varphi_x^3(t, x)}\psi_x(t, x) \qquad t \in [0, 1] \quad x \in \mathbb{S}. \tag{3.8}$$

[10] Relation (3.5) is an expression of the conservation of momentum: we refer to the end of section 4.2 for a detailed discussion of this aspect in the context of the H^1 right-invariant metric, refraining from repeating the procedure here.

In the special case $v(x) = c > 0$, $x \in \mathbb{S}$, it is easy to see by (3.2) that $\varphi(t, x) = x + ct$ and (3.8) becomes

$$\psi_t = -2c\psi_x + w \qquad t \in [0, 1] \quad x \in \mathbb{S}.$$

Since $\psi(0, x) = 0$, $x \in \mathbb{S}$, we deduce that

$$\psi(t, x) = \frac{1}{2c}\int_{x-2ct}^{x} w(y)\,\mathrm{d}y \qquad x \in \mathbb{S}$$

thus, if $v(x) = c > 0$, we have

$$(D\mathfrak{exp}(v) \cdot w)(x) = \frac{1}{2c}\int_{x-2c}^{x} w(y)\,\mathrm{d}y \qquad x \in \mathbb{S}. \tag{3.9}$$

This relation shows that, under the assumption that $\mathfrak{exp}$ is locally C^1, the derivative $D\mathfrak{exp}$ of the exponential map at $v_n(x) = \frac{1}{n}$, $x \in \mathbb{S}$, annihilates the functions $w_n(x) = \sin(\pi n x)$, $x \in \mathbb{S}$, and is therefore not invertible. This yields the desired contradiction.

To complete the proof, we have to check (3.7).

Let φ^ϵ be the geodesic on $\mathcal{D}^k$ starting at Id in the direction $(v + \epsilon w)$. Using (3.2) and (3.5) we deduce that for $\epsilon > 0$ small enough,

$$\varphi(t, x) = x + \int_0^t \frac{v(x)}{[\varphi_x(s, x)]^2}\,\mathrm{d}s \qquad t \in [0, 1] \quad x \in \mathbb{S}$$

$$\varphi^\epsilon(t, x) = x + \int_0^t \frac{v(x) + \epsilon w(x)}{\left[\varphi_x^\epsilon(s, x)\right]^2}\,\mathrm{d}s \qquad t \in [0, 1] \quad qx \in \mathbb{S}.$$

For $t \in [0, 1]$, $x \in \mathbb{S}$, we obtain

$$\begin{aligned}\frac{\varphi^\epsilon(t, x) - \varphi(t, x)}{\epsilon} &= \int_0^t \frac{w(x)}{\left[\varphi_x^\epsilon(s, x)\right]^2}\,\mathrm{d}s \\ &\quad - \int_0^t \frac{v(x)\left[\varphi_x^\epsilon(s, x) + \varphi_x(s, x)\right]}{[\varphi_x(s, x)]^2\left[\varphi_x^\epsilon(s, x)\right]^2}\,\frac{\varphi_x^\epsilon(s, x) - \varphi_x(s, x)}{\epsilon}\,\mathrm{d}s.\end{aligned} \tag{3.10}$$

We would like to let $\varepsilon \to 0$ in (3.10) and in doing so, we seek to apply the Lebesgue dominated convergence theorem.

For the pointwise convergence, by (3.6) we have

$$\varphi^\epsilon(t) - \varphi(t) = \mathfrak{exp}(t(v + \epsilon w)) - \mathfrak{exp}(tv)$$

so that,

$$\begin{cases} \lim\limits_{\epsilon\to 0} \dfrac{\varphi^\epsilon(t, x) - \varphi(t, x)}{\epsilon} = \psi(t, x) & \text{uniformly on} \quad \mathbb{S} \\ \lim\limits_{\epsilon\to 0} \dfrac{\varphi_x^\epsilon(t, x) - \varphi_x(t, x)}{\epsilon} = \psi_x(t, x) & \text{uniformly on} \quad \mathbb{S} \end{cases} \tag{3.11}$$

in view of the compact imbedding of $H^2(\mathbb{S})$ in $C^1(\mathbb{S})$.

To obtain a uniform bound under the integral sign in (3.10), we proceed as follows. Fix $t \in [0, 1]$ and $\epsilon > 0$ small. For $\epsilon \in (0, \epsilon_0)$ we define

$$F : [0, 1] \to H^k(\mathbb{S}) \qquad F(s) = \frac{\mathfrak{exp}(tv + \epsilon s w) - \mathfrak{exp}(tv)}{\epsilon} - sD\mathfrak{exp}(tv) \cdot w.$$

By the mean-value theorem and the fact that by assumption $\mathfrak{exp}$ is C^1, we infer that

$$\begin{aligned}\|F(s)\|_{H^k} &= \|F(s) - F(0)\|_{H^k} \leqslant \max_{\xi\in[0,1]} \|F'(\xi)\|_{H^k} \\ &= \max_{\xi\in[0,1]} \|D\mathfrak{exp}(tv + \epsilon\xi w) \cdot w - D\mathfrak{exp}(tv) \cdot w\|_{H^k} \leqslant M\end{aligned}$$

for some $M > 0$ that is independent of $t \in [0, 1]$ and of $\varepsilon \in (0, \epsilon_0)$. We deduce that for all $t \in [0, 1]$, $\epsilon \in (0, \epsilon_0)$,

$$\left\| \frac{\mathfrak{exp}(tv + \epsilon t w) - \mathfrak{exp}(tv)}{\epsilon} - t D\mathfrak{exp}(tv) \cdot w \right\|_{H^k} \leqslant M.$$

This relation yields

$$\sup_{t\in[0,1], x\in\mathbb{S}} \left\| \frac{\varphi_x^\epsilon(t,x) - \varphi_x(t,x)}{\epsilon} - \psi_x(t,x) \right\|_{H^k} \leqslant M \qquad \epsilon \in (0, \epsilon_0).$$

Taking into account the previous relation and (3.11) while letting $\epsilon \to 0$ in (3.10) leads to

$$\psi(t,x) = \int_0^t \frac{w(x)}{\varphi_x^2(s,x)}\,\mathrm{d}s - \int_0^t \frac{2v(x)}{\varphi_x^3(s,x)}\psi_x(s,x)\,\mathrm{d}s \qquad t \in [0,1] \quad x \in \mathbb{S}$$

in view of the Lebesgue dominated convergence theorem. The proof is complete. □

Let us now prove that the Riemannian exponential map for the (weak) L^2 right-invariant metric on $\mathcal{D}$ does not define charts.

Theorem 3. *The Riemannian exponential map of the L^2 right-invariant metric on $\mathcal{D}$ is not a C^1 diffeomorphism from a neighbourhood of zero in $T_{Id}\mathcal{D} \equiv C^\infty(\mathbb{S})$ to $\mathcal{D}$.*

Proof. Assume $\mathfrak{exp}$ is a local C^1 diffeomorphism. Note that in the proof of proposition 2 we computed directional derivatives. Take $v, w \in C^\infty(\mathbb{S})$ and fix $k \geqslant 2$. The same arguments show that $D\mathfrak{exp}(v) \cdot w$ is given precisely by (3.9) if $v(x) = c > 0, x \in \mathbb{S}$. As v_n, w_n defined above happen to belong to $C^\infty(\mathbb{S})$ with $v_n \to 0$ in $C^\infty(\mathbb{S})$, we conclude that $D\mathfrak{exp}(v_n)$ annihilates w_n and is therefore not invertible in any neighbourhood of $0 \in C^\infty(\mathbb{S})$. The obtained contradiction completes the proof. □

3.4. Breakdown of the geodesic flow

We saw that most of the geodesics have a finite lifespan $T < \infty$ given by (3.4). Let us prove that it is not possible to consider a weaker dependence on time of the geodesic that could allow us to continue each geodesic past this time $T < \infty$. Take $u_0 = -\sin(2\pi x)$, $x \in [0, 1]$. In view of (3.4), the maximal existence time of the corresponding solution $u(t, x)$ to (3.1) is $T = \frac{1}{6\pi}$. Using (3.1) it is easy to see that odd initial data yield spatially odd solutions. Differentiating (3.3) with respect to x, we get

$$u_x(t, x - 3t\sin(2\pi x)) = \frac{-2\pi\cos(2\pi x)}{1 - 6\pi t\cos(2\pi x)} \qquad x \in [0,1]$$

so that

$$\min_{x\in\mathbb{S}} u_x(t,x) = u_x(t,0) = -\frac{2\pi}{1-6\pi t} \to -\infty \qquad \text{as} \quad t \uparrow \frac{1}{6\pi}.$$

If $\varphi(t)$ is the geodesic on $\mathcal{D}$ starting at Id in the direction u_0, note that $\varphi_t = u(t, \varphi)$ leads to $\varphi_{tx} = u_x(t, \varphi) \cdot \varphi_x$. Therefore

$$\varphi_x(t,x) = \exp\left(\int_0^t u_x(s, \varphi(s,x))\,\mathrm{d}s\right) \qquad x \in [0,1] \quad t \in \left[0, \frac{1}{6\pi}\right).$$

Evaluating at $x = 0$, we obtain

$$\varphi_x(t,0) = (1 - 6\pi t)^{\frac{1}{3}} \qquad t \in \left[0, \frac{1}{6\pi}\right).$$

Indeed, $u(t, 0) = 0$ on $\left[0, \frac{1}{6\pi}\right)$, ensured by spatial oddness, forces $\varphi(t, 0) = 0$ on $\left[0, \frac{1}{6\pi}\right)$ in view of the ordinary differential equation $\frac{\mathrm{d}}{\mathrm{d}t}\varphi(t, 0) = u(t, \varphi(t, 0))$ with a locally Lipschitz right-hand side. We see that $\varphi_x(t, 0) \to 0$ as $t \uparrow \frac{1}{6\pi}$. Therefore, letting $t \uparrow T$ on the geodesic $t \mapsto \varphi(t)$, we do not obtain a $C^\infty(\mathbb{S})$ diffeomorphism in the limit.

4. The H^1 right-invariant metric

The results of the previous section raise the question whether another right-invariant metric could provide $\mathcal{D}$ with a nice local geometric structure.

We consider now the H^1 inner product on $T_{Id}\mathcal{D} \equiv C^\infty(\mathbb{S})$:

$$\langle u, v\rangle_{H^1} = \int_{\mathbb{S}} (u(x)v(x) + u'(x)v'(x))\,\mathrm{d}x \qquad u, v \in C^\infty(\mathbb{S})$$

that is moved by right translation to define a smooth right-invariant metric on $\mathcal{D}$, see section 2.2. A straightforward calculation yields

$$B(u, v) = -\left(1-\partial_x^2\right)^{-1}\left(2v_x\left(1-\partial_x^2\right)u + v\left(1-\partial_x^2\right)u_x\right) \qquad u, v \in C^\infty(\mathbb{S})$$

so that theorem 1 ensures the existence of a Riemannian connection. The geodesic equation for the H^1 right-invariant metric is

$$u_t + uu_x + \partial_x\left(1-\partial_x^2\right)^{-1}\left(u^2 + \tfrac{1}{2}u_x^2\right) = 0 \tag{4.1}$$

where $t \mapsto \varphi(t, \cdot)$ is the geodesic curve starting at time $t = 0$ at the identity Id in the direction $u_0 \in T_{Id}\mathcal{D}$ and $u = \varphi_t \in T_{\varphi(t)}\mathcal{D}$. We write this as the system

$$\begin{cases} \varphi_t = u(t, \varphi) \\ u_t + uu_x + \partial_x\left(1-\partial_x^2\right)^{-1}\left(u^2 + \tfrac{1}{2}u_x^2\right) = 0 \end{cases} \tag{4.2}$$

with initial data $\varphi(0) = Id$, $u_0 \in C^\infty(\mathbb{S})$.

4.1. The geodesic equation

Fokas and Fuchssteiner [24] obtained (4.1) as a bi-Hamiltonian abstract equation by the method of recursion operators. In dimensionless spacetime variables (x, t), (4.1) arises in several physical contexts. According to Camassa and Holm [8], it is a model for the unidirectional propagation of waves under the influence of gravity at the free surface of a shallow layer of water[11] over a flat bottom [8] with $u(t, x)$ representing the horizontal component of the velocity or, equivalently, water's free surface [9]. Equation (4.1) is a model for finite-length and small-amplitude axial-radial deformation waves in cylindrical rods composed of a compressible hyperelastic material [18] with $u(t, x)$ representing the radial stretch relative to a pre-stressed state. We would also like to point out that the viscous three-dimensional generalization of (4.1) can be used as the basis for a turbulence closure model [10] and was considered and studied in the theory of second-grade fluids [11] (examples of second-grade fluids include molten asphalt, honey, paints; the relevance of such a fluid is that it can climb up a rod which is rotating in an open vat [21]).

In the expression $\frac{1}{2}\int_{\mathbb{S}}\left(u^2 + u_x^2\right)\mathrm{d}x$, which is conserved along the flow of (4.1), the first term represents the kinetic energy induced by the horizontal component of the velocity while the second part stands for the kinetic energy due to vertical motion [27]. Since the propagation is unidirectional, the transversal horizontal motion is neglected.

Let us discuss some aspects of the partial differential equation (4.1).

The methods of [13] show that for every $u_0 \in H^k(\mathbb{S})$, $k \geqslant 2$, there exists a maximal time $T = T(u_0) > 0$ such that (4.1) has a unique solution $u \in C([0, T); H^{k-1}(\mathbb{S})) \cap C^1([0, T); H^k(\mathbb{S}))$. The solution depends continuously on the initial data in the $H^k(\mathbb{S})$ norm. Note that the conservation of the energy functional $\frac{1}{2}\int_{\mathbb{S}}\left(u^2 + u_x^2\right)\mathrm{d}x$ ensures that all solutions to (4.1) remain uniformly bounded. Moreover, the only reason that this solution fails to exist

[11] For an alternative derivation of this model in the context of water waves, we refer to [29].

for all time is that the wave breaks [14]. This means that the solution remains bounded while its slope becomes unbounded at a finite time $T > 0$. As an obvious consequence we infer that the maximal existence time does not depend on the degree of smoothness of $u_0 \in H^k(\mathbb{S})$, $k \geqslant 2$. Under some conditions the solution is global. Associate with each initial profile $u_0 \in H^k(\mathbb{S})$, $k \geqslant 2$, the expression $y_0 := u_0 - u_{0,xx}$. If y_0 does not change sign properly, the solution is global [16]. This condition is also necessary for global existence [34]. In the case of wave breaking, the rate of blow-up is given by [14]

$$\lim_{t \uparrow T} \left(\inf_{x \in \mathbb{S}} \{u_x(t,x)\}(T-t) \right) = -2.$$

For a large class of initial profiles it is also possible to determine the exact blow-up set. If $y_0 \in H^1(\mathbb{S})$ is such that $y_0(x) \leqslant 0$ for $x \in \left[0, \frac{1}{2}\right]$, y_0 is odd with $y_0 \equiv 0$ for $|x| \leqslant x_0$ with $x_0 \in \left(0, \frac{1}{2}\right)$, $y_0 \not\equiv 0$, then the blow-up set consists of the three points $\left\{0, \frac{1}{2}, 1\right\}$. More precisely, we have

$$u_x(t,0) = u_x\left(t, \tfrac{1}{2}\right) = u_x(t,1) \to -\infty \qquad \text{as} \quad t \to T < \infty$$

while (recall that u remains uniformly bounded)

$$\sup_{t \in [0,T)} |u_x(t,x)| < \infty \qquad \text{for every} \quad x \in \left(0, \tfrac{1}{2}\right) \cup \left(\tfrac{1}{2}, 1\right).$$

An interesting aspect of equation (4.1) is its integrability in the sense of the infinite-dimensional extension of Liouville's theorem for classical completely integrable Hamiltonian systems: there is a transformation which converts the equation into an infinite sequence of linear ordinary differential equations which can be trivially integrated[12]. Equation (4.1) is integrable provided the initial data u_0 are regular and the associated y_0 has no zeros—for details we refer to [15]. Let us mention that (4.1) is a counterexample to a conjecture on the complete integrability of nonlinear partial differential equations, the Painlevé test, see [25].

Equation (4.1) admits travelling wave solutions, i.e. solutions of the form $u(t,x) = \phi(x - ct)$ which travel with fixed speed c. Further, these travelling wave solutions are solitons[13]: two travelling waves reconstitute their shape and size after interacting with each other, as discovered by Camassa and Holm [8]. For a discussion of the soliton interaction for (1.2) we refer to [5]. The solitons are stable, the appropriate notion of stability being orbital stability [17]. That is, a wave starting close to a solitary wave always remains close to some translate of it at all later times. Thus the shape of the wave remains approximately the same for all times.

The fact that equation (4.1) is formally a re-expression of the geodesic flow in the group of compressible diffeomorphisms of the circle endowed with the H^1 right-invariant metric was already noted in [36]. As we will see below, the rigorous study of the geodesic flow leads to a proof of the least action principle.

It is quite natural to view (4.1) as the geodesic equation for the right-invariant H^1-metric on the Hilbert manifolds $\mathcal{D}^k$, $k \geqslant 3$. However, this needs further justification since, in contrast to the case of $\mathcal{D}$, we cannot start from the notion of covariant derivative to define the geodesics. Just as in the situation encountered in section 3.2, the alleged covariant derivative given by theorem 1 is not well defined on $\mathcal{D}^k$ due to loss of smoothness. We would also like to point out that if $X \in \mathcal{X}(\mathcal{D}^k)$, $k \geqslant 3$, then the map $\eta \mapsto X(\eta) \circ \eta^{-1}$ is only continuous on $\mathcal{D}^k$ so that the H^1 right-invariant metric on $\mathcal{D}^k$ is not smooth whereas the H^1 right-invariant metric on $\mathcal{D}$ is smooth. To fully justify why we are entitled to call (4.1) the geodesic equation on $\mathcal{D}^k$, we will show that it arises from the necessary condition for a regularly parametrized path

[12] An introduction to the ideas of integrability, coupled with a description of some examples, is provided by [33].
[13] A clear exposition of most of the essential features of soliton theory is given in [20]; see also the survey paper [39].

to be locally the shortest path on $\mathcal{D}^k$ between its fixed endpoints. In view of the comments on a similar issue made in section 3.2, we can assume the path to be parametrized by arc length, $\gamma : [0, c] \to \mathcal{D}$, and the necessary condition for γ to be locally the shortest path on $\mathcal{D}^k$ between its fixed endpoints is that γ is a critical point in the space of paths for the action functional, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\mathfrak{a}(\gamma+\epsilon\eta)\Big|_{\epsilon=0}=0$$

for every path $\eta : [0, c] \to \mathcal{D}^k$ with endpoints at zero and such that $\gamma + \epsilon\eta$ is a small variation of γ on $\mathcal{D}^k$. A lengthy calculation, similar to the one presented in section 3.2, shows that

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\mathfrak{a}(\gamma+\epsilon\eta)\Big|_{\epsilon=0}=-\int_0^c\int_{\mathbb{S}}(\eta\circ\gamma^{-1})[u_t-u_{txx}+3uu_x-2u_xu_{xx}-uu_{xxx}]\,\mathrm{d}x\,\mathrm{d}t.$$

This yields the Euler–Lagrange equation

$$u_t-u_{txx}+3uu_x-2u_xu_{xx}-uu_{xxx}=0$$

where $u = \gamma_t \circ \gamma^{-1}$ and $t \mapsto \gamma(t) \in \mathcal{D}^k$ is the curve (parametrized by arc length) yielding the critical point of the length functional to be minimized. Applying the operator $\left(1-\partial_x^2\right)^{-1}$ to the above form of the Euler–Lagrange equation we obtain (4.1). The variational formulation can therefore be used to give a meaning to (4.1) as the geodesic equation on $\mathcal{D}^k$, $k \geqslant 3$.

From the stated analytical results on (4.1) and lemma 3 we draw some first conclusions about the geodesic flow on $\mathcal{D}^k$.

Proposition 3. *For every $u_0 \in T_{Id}\mathcal{D}^k \equiv H^k(\mathbb{S})$, $k \geqslant 3$, there exists a unique geodesic on $\mathcal{D}^k$, starting at Id in the direction of u_0. Certain geodesics are defined for some finite maximal time $T > 0$ while others can be continued indefinitely in time.*

In the above result, a geodesic is a solution to (4.1) with a C^1-dependence on time, as ensured by lemma 3. As a by-product of proposition 4 below we will see that the time dependence of the geodesic is actually C^2. Note the contrast to the case of the L^2 right-invariant metric on $\mathcal{D}^k$ where the dependence cannot generally be C^2 in view of the comments preceding proposition 1.

4.2. The exponential map

We define now the Riemannian exponential map $\mathfrak{exp}$ of the H^1 right-invariant metric and study some of its properties. If $\varphi(t; u_0)$ is the geodesic on $\mathcal{D}$ or on $\mathcal{D}^k$, $k \geqslant 3$, starting at Id in the direction u_0, note that

$$\varphi(t;su_0)=\varphi(ts;u_0) \tag{4.3}$$

for $t, s \geqslant 0$ such that both sides are well defined. The continuous dependence on initial data of the solutions to (4.1) and lemma 3 show that there is some $\delta > 0$ so that all geodesics $\varphi(t; u_0)$ are defined on the same time interval $[0, T]$ with $T > 0$, provided $\|u_0\|_{H^2} < \delta$. For $\|u_0\|_{H^k} < \frac{2\delta}{T}$, we define $\mathfrak{exp}(u_0) = \varphi(1; u_0)$. In contrast to the case of the L^2 right-invariant metric, we have

Proposition 4. *The Riemannian exponential map of the H^1 right-invariant metric on $\mathcal{D}^k$, $k \geqslant 3$, is a C^1 local diffeomorphism from a neighbourhood of zero on $T_{Id}\mathcal{D}^k$ to a neighbourhood of Id on $\mathcal{D}^k$.*

Proof. We recast (4.2) as a differential system

$$\begin{cases}\varphi_t=v\\ v_t=P_\varphi(v)\end{cases}$$

where $v = u(t, \varphi)$ and the operator P_φ is given by

$$P_\varphi(v) = -\left\{\partial_x \left(1-\partial_x^2\right)^{-1}\left((v\circ\varphi^{-1})^2 + \tfrac{1}{2}(v\circ\varphi^{-1})_x^2\right)\right\}\circ\varphi.$$

Note that P_φ is a composition of the two operators

$$D_\varphi = R_\varphi \circ \partial_x \circ R_{\varphi^{-1}} \qquad Q_\varphi = R_\varphi \circ \left(1-\partial_x^2\right)^{-1}\circ R_{\varphi^{-1}}$$

with

$$E_\varphi(w) = -R_\varphi \circ \left(w^2 + \tfrac{1}{2}w_x^2\right)\circ R_{\varphi^{-1}}$$

i.e. $P_\varphi(v) = (D_\varphi \circ Q_\varphi \circ E_\varphi)(v)$ for $v \in H^k(\mathbb{S})$. We will prove that the map $(\varphi, v) \mapsto (v, P_\varphi(v))$ is C^1 from a small neighbourhood of $(Id, 0) \in \mathcal{D}^k \times H^k(\mathbb{S})$ to $H^k(\mathbb{S}) \times H^k(\mathbb{S})$. The theorem on the dependence on initial data for solutions of differential equations in Banach spaces (see [32]) ensures then that $\mathfrak{exp}$ is of class C^1. Observe that $D\mathfrak{exp}_0$ is the identity. Indeed, let $t \mapsto tv$ be a curve in $T_{Id}\mathcal{D}^k$. For $t > 0$ small enough, we have by (4.3) that $\mathfrak{exp}(tv) = \varphi(1; tv) = \varphi(t; v)$ so that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{exp}(tv)\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\varphi(t; v)\Big|_{t=0} = v \qquad v \in T_{Id}\mathcal{D}^k.$$

This shows that $D\mathfrak{exp}_0$ is the identity. Therefore, if the map $(\varphi, v) \mapsto (v, P_\varphi(v))$ is locally C^1, the assertion of proposition 4 follows from the inverse function theorem.

To complete the proof, let us show that $(\varphi, v) \mapsto (v, P_\varphi(v))$ is C^1 from a small neighbourhood of $(Id, 0) \in \mathcal{D}^k \times H^k(\mathbb{S})$ to $H^k(\mathbb{S}) \times H^k(\mathbb{S})$.

Note that the map

$$(\varphi, v) \mapsto (\varphi, E_\varphi(v)) \quad \text{is} \quad C^1 \quad \text{from} \quad \mathcal{D}^k \times H^k(\mathbb{S}) \quad \text{to} \quad \mathcal{D}^k \times H^{k-1}(\mathbb{S})$$

on a small neighbourhood of $(Id, 0)$, while

$$(\varphi, w) \mapsto (\varphi, D_\varphi(w)) \quad \text{is} \quad C^1 \quad \text{from} \quad \mathcal{D}^k \times H^{k+1}(\mathbb{S}) \quad \text{to} \quad \mathcal{D}^k \times H^k(\mathbb{S})$$

on a small neighbourhood of $(Id, 0)$, as one can see by explicit calculations. If we show that on a small neighbourhood of $(Id, 0) \in \mathcal{D}^k \times H^{k-1}(\mathbb{S})$,

$$(\varphi, w) \mapsto (\varphi, Q_\varphi(w)) \quad \text{is} \quad C^1 \quad \text{to} \quad \mathcal{D}^k \times H^{k+1}(\mathbb{S}) \tag{4.4}$$

the proof is complete. Indeed, combining the previous three assertions we infer that the map $(\varphi, v) \mapsto P_\varphi(v)$ is C^1 from a small neighbourhood of $(Id, 0) \in \mathcal{D}^k \times H^k(\mathbb{S})$ to $H^k(\mathbb{S})$. Clearly, the map $(\varphi, v) \mapsto (v, P_\varphi(v))$ will be then C^1 from a small neighbourhood of $(Id, 0) \in \mathcal{D}^k \times H^k(\mathbb{S})$ to $H^k(\mathbb{S}) \times H^k(\mathbb{S})$ and we are done.

The inverse of the map in (4.4) is the map S given by

$$(\varphi, w) \mapsto \left(\varphi, \left[R_\varphi \circ \left(1-\partial_x^2\right)\circ R_{\varphi^{-1}}\right](w)\right).$$

By explicit calculation (see below) it is easy to see that S is of class C^1 from $\mathcal{D}^k \times H^{k+1}(\mathbb{S})$ to $\mathcal{D}^k \times H^{k-1}(\mathbb{S})$. Therefore, to conclude that (4.4) holds, in view of the inverse function theorem, it will be enough to check that the Fréchet differential of S at $(Id, 0)$ is invertible. The C^1 regularity of S ensures that the Fréchet differential can be computed by calculating directional derivatives (it is actually the other way around that we showed S to be C^1). Clearly, considering partial derivatives D_i, $i = 1, 2$, of the components S_i, $i = 1, 2$, we have

$$D_1 S_1 = Id \qquad D_2 S_1 = 0$$

while

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\varepsilon}S_2(\varphi + \varepsilon\psi, w)\Big|_{\varepsilon=0} &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left\{\left(\left(1-\partial_x^2\right)\left[w\circ(\varphi+\varepsilon\psi)^{-1}\right]\right)\circ(\varphi+\varepsilon\psi)\right\}_{\varepsilon=0}\\
&= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left\{w - w_{xx}\frac{1}{(\varphi_x+\varepsilon\psi_x)^2} + w_x\frac{\varphi_{xx}+\varepsilon\psi_{xx}}{(\varphi_x+\varepsilon\psi_x)^3}\right\}_{\varepsilon=0}\\
&= \frac{2w_{xx}\psi_x}{\varphi_x^3} + \frac{w_x\psi_{xx}}{\varphi_x^3} - \frac{3\psi_x\varphi_{xx}w_x}{\varphi_x^4}.
\end{aligned}$$

On the other hand

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}S_2(\varphi, w+\varepsilon z)\bigg|_{\varepsilon=0} = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left\{(w+\varepsilon z) - (w_{xx}+\varepsilon z_{xx})\frac{1}{\varphi_x^2} + (w_x+\varepsilon z_x)\frac{\varphi_{xx}}{\varphi_x^3}\right\}_{\varepsilon=0}$$

$$= z - z_{xx}\frac{1}{\varphi_x^2} + z_x\frac{\varphi_{xx}}{\varphi_x^3}.$$

We deduce that

$$DS_{(Id,0)} = \begin{pmatrix} Id & 0 \\ 0 & 1-\partial_x^2 \end{pmatrix} \in \mathrm{Isom}(\mathcal{D}^k \times H^{k+1}(\mathbb{S}), \mathcal{D}^k \times H^{k-1}(\mathbb{S}))$$

and the proof is complete. □

In order to use the geodesic flow on the Hilbert manifolds $\mathcal{D}^k$, $k \geqslant 3$, to obtain information about the geodesic flow on $\mathcal{D}$ with the H^1 right-invariant metric, it is necessary to investigate further aspects.

Lemma 4. *Let $t \mapsto \varphi(t)$ be the geodesic on $\mathcal{D}^k$, $k \geqslant 3$, issuing from the identity in the direction of $u_0 \in H^k(\mathbb{S})$ and defined for some maximal time $T > 0$. Then $\varphi \in C^2([0,T); \mathcal{D}^k)$. If $u_0 \notin H^{k+1}(\mathbb{S})$, for all times $t \in (0,T)$ we have $\varphi(t) \notin H^{k+1}(\mathbb{S})$.*

Proof. By proposition 4 we know that there exists a unique geodesic φ on $\mathcal{D}^k$. The C^2-dependence on time of the geodesic follows from the recasting of (4.2) as a differential system with a C^1 right-hand side, cf the proof of proposition 4.

If $u \in C^1([0,T); H^{k-1}(\mathbb{S})) \cap C([0,T); H^k(\mathbb{S}))$ is the solution of (4.1) with initial data u_0, let $m = u - u_{xx}$. Using (4.1), it is easy to check (differentiating with respect to time) that the following identity holds[14]:

$$m(t, \varphi(t,x)) \cdot \varphi_x^2(t,x) = m_0(x) \qquad t \in [0,T).$$

Using the previous identity and (4.2), a straightforward calculation shows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\varphi_{xx}}{\varphi_x} = u\circ\varphi\cdot\varphi_x - \frac{m_0}{\varphi_x} \qquad t \in (0,T).$$

Hence

$$\frac{\varphi_{xx}(t)}{\varphi_x(t)} = \int_0^t u(s,\cdot)\circ\varphi(s)\cdot\varphi_x(s)\,\mathrm{d}s - m_0\int_0^t \frac{1}{\varphi_x(s)}\,\mathrm{d}s \qquad t \in [0,T). \tag{4.5}$$

Assume now that for some $u_0 \notin H^{k+1}(\mathbb{S})$ we have $\varphi(t) \in H^{k+1}(\mathbb{S})$ at some time $t \in (0,T)$. Observe now that $u\circ\varphi \in C([0,t]; H^k(\mathbb{S}))$ and $\frac{1}{\varphi_x} \in C([0,t]; H^{k-1}(\mathbb{S}))$. Therefore, relation (4.5) would force $m_0 \in H^{k-1}(\mathbb{S})$, that is, $u_0 \in H^{k+1}(\mathbb{S})$. The obtained contradiction completes the proof. □

We are now in a position to prove the following result.

Theorem 4. *Let $u_0 \in T_{Id}\mathcal{D} \equiv C^\infty(\mathbb{S})$. There exists a unique geodesic $\varphi \in C^2([0,T); \mathcal{D})$ starting at Id in the direction u_0; the geodesic is defined for all times if and only if $(u_0 - u_0'')$ does not change sign properly.*

Proof. Fix $k \geqslant 3$. In view of lemma 4, there exists a unique geodesic $\varphi \in C^2([0,T); \mathcal{D}^k)$. The assertion about T and the fact that $\varphi \in \mathcal{D}$ are consequences of the properties of (4.1)—the fact that a singularity can arise in a solution only in the form of wave breaking allows us to deduce that if $u_0 \in C^\infty(\mathbb{S})$, then the unique solution of (4.1) with data u_0 is smooth on its

[14] Following the proof of theorem 5 there is a discussion of the origins of this identity.

maximal interval of existence. Also, since $\varphi \in C^2([0, T); \mathcal{D}^k)$ for all $k \geqslant 2$, we see that $\varphi \in C^2([0, T); \mathcal{D})$, due to the way differentiation and convergence are defined on $C^\infty(\mathbb{S})$. □

Let us now analyse whether the $\mathfrak{exp}$ is a local C^1 diffeomorphism for $\mathcal{D}$. We would like to point out that, unlike the case of the Hilbert manifolds $\mathcal{D}^k$, $k \geqslant 3$, we cannot apply the inverse function theorem as we deal with a Fréchet manifold. Even under the assumption that $P : U \subset F \to V \subset G$ defines a smooth map between open sets in Fréchet spaces such that for every $f \in U$ the derivative $DP(f)$ is an invertible linear map of F to G with a smooth inverse $(DP)^{-1} : U \times G \subset F \times G \to F$ (we avoid spaces of linear maps), local invertibility of P is not ensured, see [26], p 125. The use of the Nash–Moser theorem would require special properties of the maps under discussion [26]. It is at this point that the use of the information on the geodesic flow of $\mathcal{D}^k$ will yield a relatively simple proof of the following result that marks the striking difference between the L^2 and H^1 right-invariant metrics on $\mathcal{D}$, see theorem 3.

Theorem 5. *The Riemannian exponential map of the H^1 right-invariant metric on $\mathcal{D}$ is a C^1 diffeomorphism from a neighbourhood of zero in $T_{Id}\mathcal{D} \equiv C^\infty(\mathbb{S})$ to a neighbourhood of Id on $\mathcal{D}$.*

Proof. From proposition 4 we know that $\mathfrak{exp}$ is a C^1 diffeomorphism from an open neighbourhood U_3 of $0 \in H^3(\mathbb{S})$ to an open neighbourhood V_3 of Id on $\mathcal{D}^3$; we can take U_3 such that at every point of U_3, the differential of $\mathfrak{exp}$ is a bijection of $H^3(\mathbb{S})$. Observe that $U = U_3 \cap C^\infty(\mathbb{S})$ and $V = V_3 \cap C^\infty(\mathbb{S})$ are open neighbourhoods of $0 \in C^\infty(\mathbb{S})$, respectively $Id \in \mathcal{D}$. Theorem 4 ensures that $\mathfrak{exp}(U) \subset V$. On the other hand, we know from lemma 4 that if $\mathfrak{exp}(u_0) \in V$ for some $u_0 \in U_3$, then necessarily $u_0 \in U$. Therefore $\mathfrak{exp}$ is a local bijection from U to V.

We will prove now that $\mathfrak{exp}$ is a C^1 diffeomorphism from U to V. Recall that convergence in $C^\infty(\mathbb{S})$ means convergence in all $H^m(\mathbb{S})$ spaces for all m large enough.

Let $u_0 \in U$. The proof of proposition 4 shows that $\mathfrak{exp}$ is a C^1-map on every $U_3 \cap H^k(\mathbb{S}), k \geqslant 3$, so that $D\mathfrak{exp}_{u_0}$ is a bounded linear operator from $H^k(\mathbb{S})$ to $H^k(\mathbb{S})$. We will prove that $D\mathfrak{exp}_{u_0}$ is a bijection from $H^k(\mathbb{S})$ to $H^k(\mathbb{S})$ for all $k \geqslant 3$. Then, in view of the inverse function theorem, both $\mathfrak{exp}$ and its inverse are C^1-maps on small $H^k(\mathbb{S})$-neighbourhoods of $u_0 \in U$, respectively, $\mathfrak{exp}(u_0) \in V$. As $k \geqslant 3$ is arbitrary, we would infer that $\mathfrak{exp}$ is a C^1 diffeomorphism from U to V.

To prove the last step, we use an inductive argument. To start with, $D\mathfrak{exp}_{u_0}$ is a bijection from $H^3(\mathbb{S})$ to $H^3(\mathbb{S})$ as $u_0 \in U_3$. Fix $k \geqslant 3$, assume that for $j = 3, \ldots, k$ the map $D\mathfrak{exp}_{u_0}$ is a bijection from $H^j(\mathbb{S})$ to $H^j(\mathbb{S})$ and let us show that $D\mathfrak{exp}_{u_0}$ is a bijection from $H^{k+1}(\mathbb{S})$ to $H^{k+1}(\mathbb{S})$. Clearly, $D\mathfrak{exp}_{u_0}$ is injective as a bounded linear map from $H^{k+1}(\mathbb{S})$ to $H^{k+1}(\mathbb{S})$ since its extension to $H^k(\mathbb{S})$ is injective. To show that it is surjective, it suffices to prove that there is no $v \in H^k(\mathbb{S}), v \notin H^{k+1}(\mathbb{S})$, with $D\mathfrak{exp}_{u_0}(v) \in H^{k+1}$. Indeed, $D\mathfrak{exp}_{u_0}(H^{k+1}(\mathbb{S})) \subset H^{k+1}(\mathbb{S})$ by the fact that $\mathfrak{exp}$ is a C^1-map on $U_3 \cap H^{k+1}(\mathbb{S})$, while $D\mathfrak{exp}_{u_0}(H^k(\mathbb{S})) = H^k(\mathbb{S})$ by the inductive assumption. If $v \in H^k(\mathbb{S}), v \notin H^{k+1}(\mathbb{S})$, is such that $D\mathfrak{exp}_{u_0}(v) \in H^{k+1}$, for $\varepsilon > 0$ small enough let $\varphi^\varepsilon(t)$ be the geodesic on $\mathcal{D}^k(\mathbb{S})$ from Id in the direction $u_0 + \varepsilon v$. If $u^\varepsilon \in C^1([0, T^\varepsilon); H^{k-1}(\mathbb{S})) \cap C([0, T^\varepsilon); H^k(\mathbb{S}))$ is the solution of (4.1) with initial data $u_0 + \varepsilon v$, the proof of proposition 4 ensures that the map $(\varphi^\varepsilon(t), u^\varepsilon(t)) \in \mathcal{D}^k \times H^k(\mathbb{S})$ has a C^1-dependence on ε and $t \in [0, 1]$. From (4.5) we obtain

$$\frac{\varphi^\varepsilon_{xx}(1)}{\varphi^\varepsilon_x(1)} = \int_0^1 u^\varepsilon(t, \cdot) \circ \varphi^\varepsilon(t) \cdot \varphi^\varepsilon_x(t)\,\mathrm{d}t - (u_0 - u_{0,xx} + \varepsilon v - \varepsilon v_{xx}) \int_0^1 \frac{1}{\varphi^\varepsilon_x(t)}\,\mathrm{d}t.$$

Multiplying both sides with $\varphi_x^\varepsilon(1)$ and differentiating afterwards with respect to ε as the derivative of an integral depending on a parameter in Banach spaces [19], a calculation shows that

$$D\mathfrak{exp}_{u_0}(v) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\varphi^\varepsilon(1)\bigg|_{\varepsilon=0} \in H^{k+1}(\mathbb{S})$$

is possible only if[15] we have $(v - v_{xx}) \in H^{k-1}(\mathbb{S})$, i.e. $v \in H^{k+1}(\mathbb{S})$. The obtained contradiction permits us to conclude. □

We would like to comment on the relation $(*)$ that plays a crucial role in the approach. For finite-dimensional Lie groups the geodesic flow of a one-sided invariant metric has as a remarkable conservation law the angular momentum [2]. This, in view of Noether's theorem, is a consequence of the invariance of the metric by the action of the group on itself. The same conclusion can be drawn by formally carrying over the reasoning to infinite-dimensional Lie groups [3]. In the present case, the formal conclusion can be justified rigorously: relation $(*)$ is an expression of the conservation of momentum. More precisely, any $v \in C^\infty(\mathbb{S}) \equiv T_{Id}\mathcal{D}$ defines a one-parameter group of diffeomorphisms $h^s : \mathcal{D} \to \mathcal{D}$, $h^s(\varphi) = \varphi \circ \exp_L(sv)$, where $\exp_L$ is the Lie-group exponential map, cf section 2.1. Since the metric is by construction invariant under the action of h^s, Noether's theorem ensures that

$$\frac{\partial L}{\partial \varphi_t}(\varphi, \varphi_t)\left[\frac{\mathrm{d}h^s(\varphi)}{\mathrm{d}s}\bigg|_{s=0}\right]$$

is preserved along the geodesic curve $t \mapsto \varphi(t)$ with $\varphi(0) = Id$ and $\varphi_t(0) = u_0 \in T_{Id}\mathcal{D}$; here $L : T\mathcal{D} \to \mathbb{R}$ is the right-invariant metric. We compute

$$\frac{\mathrm{d}h^s(\varphi)}{\mathrm{d}s}\bigg|_{s=0} = \varphi_x \cdot v \qquad \frac{\partial L}{\partial v}(\varphi, v)[w] = 2\langle v \circ \varphi^{-1}, w \circ \varphi^{-1}\rangle$$

obtaining that

$$\langle \varphi_t \circ \varphi^{-1}, \varphi_x \circ \varphi^{-1} \cdot v \circ \varphi^{-1}\rangle = \langle u_0, v\rangle \qquad v \in C^\infty(\mathbb{S}).$$

Using the explicit form of our metric, and observing that $\langle f, g\rangle_{H^1} = \langle f - f_{xx}, g\rangle_{L^2}$, the above relation takes the form

$$\int_{\mathbb{S}} (u - u_{xx}) \cdot \varphi_x \circ \varphi^{-1} \cdot v \circ \varphi^{-1}\,\mathrm{d}x = \int_{\mathbb{S}} (u_0 - u_{0,xx}) \cdot v\,\mathrm{d}x$$

where, as before, $u = \varphi_t \circ \varphi^{-1}$. Changing variables in the first integral, we obtain

$$\int_{\mathbb{S}} (u - u_{xx}) \circ \varphi \cdot \varphi_x^2 \cdot v\,\mathrm{d}x = \int_{\mathbb{S}} (u_0 - u_{0,xx}) \cdot v\,\mathrm{d}x \qquad v \in C^\infty(\mathbb{S})$$

and $(*)$ is now plain[16].

4.3. Length-minimizing property

We would now like to prove the minimizing property of the geodesics on $\mathcal{D}$ endowed with the H^1 right-invariant metric. For this we first consider some preliminary results.

The parallel transport of a vector V_0 tangent to a C^2-curve $\alpha : J \to \mathcal{D}$ at $\alpha(0) = \alpha_0$, along the curve α is defined as a curve $\gamma \in \mathrm{Lift}(\alpha)$ with $\gamma(0) = V_0$ and such that $D_{\alpha_t}\gamma \equiv 0$ on J. Let us prove

[15] The point is that in this calculation we have terms that will clearly belong to $H^{k-1}(\mathbb{S})$ and an additional term involving $(v - v_{xx})$ multiplied with the $H^{k-1}(\mathbb{S})$-function $\varphi_x(1) \cdot \int_0^1 \frac{1}{\varphi_x(t)}\,\mathrm{d}t$, where $\varphi(t)$ is the geodesic issuing from Id in the direction u_0. The factor of $(v - v_{xx})$ has no zeros.

[16] At this point it becomes clear that relation (3.5), playing a crucial role in section 3, can be justified analogously.

Proposition 5. *Let $\alpha : J \to \mathcal{D}$ be a C^2 curve defined on some open interval $J \subset \mathbb{R}$ containing zero. Given $V_0 \in T_{\alpha_0}\mathcal{D}, \alpha_0 = \alpha(0) \in \mathcal{D}$, there exists a unique lift $\gamma : J \to T\mathcal{D}$ which is α-parallel and such that $\gamma(0) = V_0$. Furthermore, parallel transport is a metric isomorphism: if γ_1, γ_2 are the unique α-parallel lifts of α with $\gamma_i(0) = V_i \in T_{\alpha_0}\mathcal{D}, i = 1, 2$, then*

$$\langle \gamma_1(t), \gamma_2(t)\rangle = \langle V_1, V_2\rangle \qquad t \in J.$$

Proof. Using the exact expression we have for B in (2.11), after some manipulations we see that the equation of parallel transport is

$$v_t - uv_x - \partial_x \left(1 - \partial_x^2\right)^{-1} \left(uv + \tfrac{1}{2}u_x v_x\right) = 0 \tag{4.6}$$

with the notation from section 2.4. It is useful to observe that for a fixed $u \in C^1(J; \mathcal{D})$, the map

$$v \mapsto \partial_x \left(1 - \partial_x^2\right)^{-1} \left(uv + \tfrac{1}{2}u_x v_x\right)$$

is a bounded linear operator from $H^k(\mathbb{S})$ to $H^k(\mathbb{S}), k \geqslant 3$. Viewing (4.6) as linear hyperbolic evolution equation in v with fixed $u \in C^1(J; \mathcal{D})$, it is known (see [31]) that, given $V_0 \in H^k(\mathbb{S}), k \geqslant 3$, there exists a unique solution

$$v \in C(J; H^k(\mathbb{S})) \cap C^1(J; H^{k-1}(\mathbb{S}))$$

of (4.6) with initial data $v(0) = V_0$. Taking into account the definition of differentiability of $C^\infty(\mathbb{S})$, letting $k \uparrow \infty$, we infer that, given $V_0 \circ \alpha_0^{-1} \in T_{Id}\mathcal{D} \equiv C^\infty(\mathbb{S})$, there exists a unique solution $v \in C^1(J; \mathcal{D})$ to (4.6) with $v(0) = V_0 \circ \alpha_0^{-1}$.

From (2.9) we deduce that $\langle \gamma_1(t), \gamma_2(t)\rangle$ is constant for any α-parallel lifts, whence the second assertion follows. □

Let $\mathcal{W}, \mathcal{U}$ be open neighbourhoods of $0 \in C^\infty(\mathbb{S})$, respectively of $Id \in \mathcal{D}$, such that the Riemannian exponential map $\mathfrak{exp}$ of the H^1 right-invariant metric on $\mathcal{D}$ is a C^1 diffeomorphism from $\mathcal{W}$ onto $\mathcal{U}$, cf theorem 5. Note that the map

$$G : \mathcal{D} \times \mathcal{W} \to \mathcal{D} \times \mathcal{D} \qquad (\eta, u) \mapsto (\eta, R_\eta \mathfrak{exp}(u))$$

is a C^1 diffeomorphism onto its image. We now define polar coordinates around $\eta \in \mathcal{D}$. Let $\mathcal{U}(\eta) = R_\eta\mathcal{U} = R_\eta \mathfrak{exp}(\mathcal{W})$. If $\varphi \in \mathcal{U}(\eta) - \{\eta\}$, then $\varphi = \mathfrak{exp}(v) \circ \eta$ for some $v \in \mathcal{W}$. We can write $v = rw$, where $\langle w, w\rangle = 1$ and $r \in \mathbb{R}_+$; (r, w) are the polar coordinates of $\varphi \in \mathcal{U}(\eta)$.

If $J_1, J_2 \subset \mathbb{R}$ are open intervals and $\sigma : J_1 \times J_2 \to \mathcal{D}$ is a map such that $\frac{\partial^2\sigma}{\partial r^2}$ and $\frac{\partial^2\sigma}{\partial r\partial t}$ are continuous, then for every fixed $t \in J_2$ we obtain a curve $\sigma(\cdot, t) : J_1 \to \mathcal{D}$. We denote by $\partial_1\sigma$ the partial derivative with respect to r and define similarly $\partial_2\sigma$. Note that for each $t \in J_2$, the curves $r \mapsto \partial_1\sigma(r, t)$ and $r \mapsto \partial_2\sigma(r, t)$ are lifts of $r \mapsto \sigma(r, t)$. Generally, if γ is a lift of $r \mapsto \sigma(r, t)$, we may apply the covariant derivative with respect to functions of the first variable r,

$$\left(D_{\partial_1\sigma}\gamma\right)(r) = (D_1\gamma)(r, t).$$

We define $D_2\gamma$ similarly. The next lemma is the analogue of the commutator rule of partial derivatives in the context of covariant derivatives.

Lemma 5. *Let $\sigma : J_1 \times J_2 \to \mathcal{D}$ ($J_1, J_2 \subset \mathbb{R}$ are open intervals) be such that $\partial_1\partial_2\sigma$ and $\partial_2\partial_1\sigma$ exist and are continuous. Then*

$$D_1\partial_2\sigma = D_2\partial_1\sigma \tag{4.7}$$

$$\partial_2\langle\partial_1\sigma, \partial_1\sigma\rangle = 2\langle D_1\partial_2\sigma, \partial_1\sigma\rangle. \tag{4.8}$$

Proof. In a local chart we have by (2.8) that

$$D_1\partial_2\sigma = \partial_1\partial_2\sigma - Q(\partial_1\sigma, \partial_2\sigma) = D_2\partial_1\sigma$$

as Q is symmetric. This proves (4.7).

On the other hand, from lemma 2,

$$\partial_2\langle\partial_1\sigma, \partial_1\sigma\rangle = 2\langle D_2\partial_1\sigma, , \partial_1\sigma\rangle.$$

Using (4.7) we obtain relation (4.8). □

Lemma 6. *Let $\gamma : [a, b] \to \mathcal{U}(\eta) - \{\eta\}$ be a piecewise C^1-curve. Then the length of the curve is estimated by*

$$l(\gamma) \geqslant |r(b) - r(a)|$$

where $(r(t), w(t))$ are the polar coordinates of $\gamma(t)$. Equality holds if and only if the function $t \mapsto r(t)$ is monotone and the map $t \mapsto w(t) \in \mathcal{W}$ is constant.

Proof. Breaking γ up into pieces that are C^1, we may assume without loss of generality that γ itself is C^1. Also, taking into account the right-invariance of the metric, we may set $\eta = Id$. The vector $r(t)w(t)$ is obtained in a chart by the inversion of $\mathfrak{exp}$ followed by a projection so that the functions $t \mapsto r(t)$ and $t \mapsto w(t)$ are C^1.

Let $\sigma(r, t) = \mathfrak{exp}(rw(t))$ and $\gamma(t) = \sigma(r(t), t)$, where $(r(t), w(t))$ are the polar coordinates of $\gamma(t)$ in $\mathcal{U}$. In our argumentation, we will need $\frac{\partial^2\sigma}{\partial r^2}$, $\frac{\partial^2\sigma}{\partial r\partial t}$ and $\frac{\partial^2\sigma}{\partial t\partial r}$ to be continuous[17]. To prove that this holds, we first fix $k \geqslant 3$, show that the hypothesis is fulfilled in the $H^k(\mathbb{S})$-setting and then let $k \uparrow \infty$. If $\varphi(s; z)$ is the geodesic on $\mathcal{D}^k$ starting at Id in the direction $z \in H^k(\mathbb{S})$, observe that $\sigma(r, t) = \varphi(r; w(t))$ in view of (4.3). From the proof of proposition 4 we know that $\varphi(s; z)$ has a C^2-dependence on s and (φ, φ_s) has a C^1-dependence on z. This implies at once the continuity of $\frac{\partial^2\sigma}{\partial r^2}$ and $\frac{\partial^2\sigma}{\partial t\partial r}$ in the $H^k(\mathbb{S})$-setting. Furthermore, using the formula for the derivative of a Banach-valued integral depending on a parameter [19], in view of

$$\varphi(s; z) = Id + \int_0^s \frac{\partial\varphi}{\partial s}(\xi; z)\,\mathrm{d}\xi \qquad \text{in} \quad H^k(\mathbb{S})$$

we have

$$\frac{\partial\varphi}{\partial z}(s, z) = \int_0^s \frac{\partial^2\varphi}{\partial z\partial s}(\xi; z)\,\mathrm{d}\xi \qquad \text{in} \quad \mathcal{L}(H^k(\mathbb{S}), H^k(\mathbb{S}))$$

thus $\frac{\partial^2\varphi}{\partial z\partial s} = \frac{\partial^2\varphi}{\partial s\partial z}$. As $t \mapsto w(t) \in H^k(\mathbb{S})$ is a C^1-map, we obtain that $\frac{\partial^2\sigma}{\partial r\partial t}$ is continuous in the $H^k(\mathbb{S})$-setting. According to the previous comments, this intermediate step is now justified.

We proceed with the obvious relation

$$\gamma'(t) = \frac{\partial\sigma}{\partial r} \cdot r'(t) + \frac{\partial\sigma}{\partial t} \qquad t \in J. \tag{4.9}$$

On the other hand, note that $r \mapsto \sigma(r, t)$ is a geodesic so that by proposition 5 we obtain

$$\left\langle\frac{\partial\sigma}{\partial r}, \frac{\partial\sigma}{\partial r}\right\rangle = \langle w(t), w(t)\rangle \equiv 1. \tag{4.10}$$

We will now show that

$$\left\langle\frac{\partial\sigma}{\partial r}, \frac{\partial\sigma}{\partial t}\right\rangle \equiv 0. \tag{4.11}$$

[17] We only need the continuity of $\frac{\partial\sigma}{\partial t}$ and not even the existence of $\frac{\partial^2\sigma}{\partial t^2}$.

Indeed, observe that because $r \mapsto \sigma(r,t)$ is a geodesic, $(D_1 \frac{\partial\sigma}{\partial r}) = 0$. From (4.8) and (4.10) we obtain that

$$\left\langle D_1 \frac{\partial\sigma}{\partial t}, \frac{\partial\sigma}{\partial r} \right\rangle = \frac{1}{2}\partial_t \left\langle \frac{\partial\sigma}{\partial r}, \frac{\partial\sigma}{\partial r} \right\rangle \equiv 0.$$

Thus, in view of lemma 2,

$$\partial_r \left\langle \frac{\partial\sigma}{\partial r}, \frac{\partial\sigma}{\partial t} \right\rangle = \left\langle D_1 \frac{\partial\sigma}{\partial r}, \frac{\partial\sigma}{\partial t} \right\rangle + \left\langle \frac{\partial\sigma}{\partial r}, D_1 \frac{\partial\sigma}{\partial t} \right\rangle \equiv 0.$$

Therefore

$$\left\langle \frac{\partial\sigma}{\partial r}, \frac{\partial\sigma}{\partial t} \right\rangle (r,t) = \left\langle \frac{\partial\sigma}{\partial r}, \frac{\partial\sigma}{\partial t} \right\rangle (0,t).$$

But $\sigma(0,t) = Id$ thus $\frac{\partial\sigma}{\partial r}(0,t) = 0$ and (4.11) follows at once.

Combining (4.9)–(4.11), we get

$$\|\gamma'(t)\|^2 = |r'(t)|^2 + \left\| \frac{\partial\sigma}{\partial t} \right\|^2 \geqslant |r'(t)|^2 \qquad t \in [a,b]$$

so that

$$l(\gamma) \geqslant \int_a^b |r'(t)|\,\mathrm{d}t \geqslant |r(b) - r(a)|.$$

It is also immediate to infer from the above the characterization of the situation when equality holds. Indeed, $\left\| \frac{\partial\sigma}{\partial t} \right\| \equiv 0$ forces $w'(t) = 0$ since $D\mathfrak{exp}_{rw(t)}$ is a bijection if viewed as a linear map from $H^3(\mathbb{S})$ to $H^3(\mathbb{S})$. □

Let us now prove

Theorem 6. *If $\eta, \varphi \in \mathcal{D}$ are close enough, more precisely, if $\varphi \circ \eta^{-1} \in \mathcal{U}$, then η and φ can be joined by a unique geodesic in $\mathcal{U}(\eta)$. This unique geodesic is length minimizing among all piecewise C^1-curves joining η to φ on $\mathcal{D}$.*

Proof. The first assertion is a consequence of theorem 5. Indeed, if $v = \mathfrak{exp}^{-1}(\varphi \circ \eta^{-1})$, then $\alpha(t) = \mathfrak{exp}(tv) \circ \eta$ is the unique geodesic joining η to φ in $\mathcal{U}(\eta)$.

To prove the second statement, let $\varphi \circ \eta^{-1} = \mathfrak{exp}(rw)$ with $\|w\| = 1$ and choose some $\varepsilon \in (0,r)$. If γ is any piecewise C^1-curve joining η to φ on $\mathcal{D}$, then γ contains an arc of curve γ^* such that, after reparametrization,

$$\|\mathfrak{exp}^{-1}(\gamma^*(0))\| = \varepsilon \qquad \|\mathfrak{exp}^{-1}(\gamma^*(1))\| = r$$

and

$$\varepsilon \leqslant \|\mathfrak{exp}^{-1}(\gamma^*(t))\| \leqslant r \qquad t \in [0,1].$$

From lemma 6 we deduce that $\mathfrak{l}(\gamma^*) \geqslant r - \varepsilon$, thus $\mathfrak{l}(\gamma) \geqslant \mathfrak{l}(\gamma^*) \geqslant r - \varepsilon$. By the arbitrariness of $\varepsilon > 0$ we infer that $\mathfrak{l}(\gamma) \geqslant r$. But lemma 6 shows that $\mathfrak{l}(\alpha) = r$ and the minimum is attained if and only if the curve is a reparametrization of a geodesic. The proof is complete. □

We showed that a geodesic is locally the shortest path between two nearby points of $\mathcal{D}$. The geometric analysis provided a rather simple solution to the corresponding variational problem on $\mathcal{D}$. The discussion in the introduction shows that the least action principle holds, the physical interpretation being that a configuration of the system can be transformed to any nearby configuration by a unique flow of (1.2). Of all possible paths joining these two configurations, the system selects that of minimal action.

4.4. Breakdown of the geodesic flow

In special directions, the breakdown of the geodesic can be better understood. Let us associate with each solution of (4.1) the function $m = u - u_{xx}$, representing the momentum in the physical derivation of the equation [9]. If $t \mapsto \varphi(t) \in \mathcal{D}$ is the geodesic starting at Id in the direction $u_0 \in C^\infty(\mathbb{S})$, defined for some maximal time $T > 0$, recall that

$$m(t, \varphi(t, x)) \cdot \varphi_x^2(t, x) = m_0(x) \qquad t \in [0, T) \quad x \in \mathbb{S} \tag{4.12}$$

cf the proof of lemma 4. We have

Proposition 6. *Assume $u_0 \in C^\infty(\mathbb{S})$, $u_0 \not\equiv 0$, is odd and such that the corresponding m_0 satisfies $m_0(x) = 0$ for all $x \in [-x_0, x_0]$ for some $x_0 \in \left(0, \frac{1}{2}\right)$ with $m_0(x) \leqslant 0$ for $x \in \left[0, \frac{1}{2}\right]$. Then the geodesic in $\mathcal{D}$ issuing from Id in the direction u_0 breaks down in finite time $T < \infty$. In the limit $t \uparrow T$, the diffeomorphisms $\varphi(t, x)$ on this geodesic flatten out.*

Proof. Let $T > 0$ be the maximal existence time of the solution to (4.1) with initial data u_0. The special assumptions ensure that $T < \infty$, the blow-up set consists precisely of the three points $\left\{0, \frac{1}{2}, 1\right\} \in [0, 1] \equiv \mathbb{S}$,

$$u_x(t, 0) = u_x\left(t, \tfrac{1}{2}\right) = u_x(t, 1) \to -\infty \qquad \text{as} \quad t \to T < \infty$$

while u remains uniformly bounded, see the discussion in section 4.1.

Since a spatially odd solution to (4.1) remains spatially odd (see [13]) on the time-interval $[0, T)$, m remains spatially odd on $[0, T)$. Relation (4.12) proves not only that $m(t, x)$ is odd in x but also that it remains nonpositive for $x \in \left[0, \frac{1}{2}\right]$ as long as $t \in [0, T)$.

Setting $x = 0$ in $\varphi_t = u(t, \varphi)$ we see that $\varphi(t, 0) = 0$ as $t \in [0, T)$ by the uniqueness theorem for ODEs with a locally Lipschitz right-hand side.

Since $\varphi(t, 0) = 0$ for $t \in [0, T)$ and $\varphi(t, \cdot)$ is orientation preserving, we have that $\varphi(t, x) > 0$ for $(t, x) \in [0, T) \times \left(0, \frac{1}{2}\right)$. On the other hand, $m(t, x) \leqslant 0$ on $\left[0, \frac{1}{2}\right]$ and $u(t, 0) = u\left(t, \frac{1}{2}\right) = 0$ on $[0, T)$ by the oddness and the periodicity properties. Therefore the maximum principle ensures $u(t, y) \leqslant 0$ on $[0, T) \times \left[0, \frac{1}{2}\right]$. We deduce from $\varphi_t = u(t, \varphi)$ that at every fixed $x \in \left[0, \frac{1}{2}\right]$, $\varphi(t, x)$ decreases by nonnegative values as $t \uparrow T$. Therefore $\lim_{t\uparrow T} \varphi(t, x)$ exists and is nonnegative for every $x \in \left[0, \frac{1}{2}\right]$. If $y \in [0, x]$, we have by the orientation-preserving property that $0 \leqslant \varphi(t, y) \leqslant \varphi(t, x)$ as $t \in [0, T)$ so that $\lim_{t\uparrow T} \varphi(t, x) = 0$ would imply $\lim_{t\uparrow T} \varphi(t, y) = 0$ for all $y \in [0, x]$.

The previous observations show that in order to prove that $\varphi(t, x)$ flattens out in the limit $t \uparrow T$, it is enough to prove that for some $x \in \left(0, \frac{1}{2}\right]$ we have $\lim_{t\uparrow T} \varphi(t, x) = 0$.

Assume the contrary. We would have that

$$\varphi(t, x_0) \geqslant \lim_{t\uparrow T} \varphi(t, x_0) = \epsilon > 0 \qquad t \in [0, T). \tag{4.13}$$

Relation (4.12) shows that $m(t, \varphi(t, x)) = 0$ for all $(t, x) \in [0, T) \times [0, x_0]$. That is, $m(t, y) = 0$ on $[0, \varphi(t, x_0)]$ for every $t \in [0, T)$. Combining this with (4.13), we obtain by the spatial oddness of m that

$$m(t, y) = 0 \qquad (t, y) \in [0, T) \times [-\epsilon, \epsilon]. \tag{4.14}$$

Under these circumstances we do not have that $u_x(t, 0) \to -\infty$ as $t \uparrow T$. Indeed, from (4.14) and the uniform bound we have on $u(t, x)$ for $(t, x) \in [0, T) \times \mathbb{S}$ we can infer an uniform bound on $u_{xx}(t, x)$ for $(t, x) \in [0, T) \times [-\epsilon, \epsilon]$). But if a C^2-function and its second derivative are uniformly bounded on an interval, a Taylor expansion shows that the first derivative will also be uniformly bounded. On the other hand, $u_x(t, 0) \to -\infty$ as $t \uparrow T$ is exactly what happens! The obtained contradiction proves that $\varphi(t, x)$ flattens out in the limit $t \uparrow T$. □

5. Conclusion

The idea of studying geodesic flow in order to analyse the motion of inertial continuum mechanical systems is due to Arnold [1]. For discussions of this aspect for the Euler equation of an ideal fluid we refer to [23], while the geodesic property for certain ideal geophysical fluid flows is presented in [42], see also [38] for the motion of an ideal magnetohydrodynamical fluid. This approach has the appealing feature that it represents the Lagrangian formulation of the mechanical problem. However, the conceptual and technical problems of global analysis that arise are very intricate so that the approach is mostly limited to recasting the equation of motion into the form of geodesics on certain infinite-dimensional groups and obtaining results about the geodesic flow on the configuration space [1, 38, 42] by arguments that are rather heuristic in character [2, 3]. In the case of Euler's equation in hydrodynamics, progress in the direction of the geometric approach was made by enlarging the configuration space to spaces with a more convenient structure and analysing on these spaces related aspects that are of relevance in the study of the motion of an ideal fluid [22]. The results obtained so far for the actual configuration space have a formal character in view of the serious analytical difficulties encountered: a rigorous passage in this case from the enlarged configuration space to the group of smooth diffeomorphisms remains an open question, for a review of the state of the art see [27].

In this paper the one-dimensional compressible analogue of the description of the Euler equation for a perfect fluid by means of geodesic flow is considered. The fact that we deal with a one-dimensional problem makes it possible to provide a rather in-depth study of the qualitative structure of the geodesic equation. We perform a study of the geodesic motion on the configuration space (since we deal with a spatially periodic problem, the configuration space is the group of orientation-preserving diffeomorphisms of the circle—we exclude discontinuities and fluid interpenetration) leading to results about the model in mathematical physics that is under investigation. We prove that a state of the system is transformed to another nearby state by going through a uniquely determined flow that minimizes the energy and analyse the breakdown of the geodesic flow. To the best of our knowledge, the question of whether the least action principle holds in the configuration space (of volume-preserving smooth diffeomorphisms cf section 1) for the Euler equation of hydrodynamics is a question that still remains open[18]. In this context, we admit that it is worth having a model in which the geometric approach proves to be a powerful tool for studying rigorously the infinite-dimensional configuration space of the underlying hydrodynamical problem. Our results show that such a model is provided by the H^1 right-invariant metric (and not by the L^2 right-invariant metric) on the group of orientation-preserving diffeomorphisms of the circle.

Regarding possible extensions of this work, let us first note that, if instead of considering the periodic motion of (1.2), we are interested in solutions that vanish at infinity, to endow the corresponding configuration space of diffeomorphisms of the line with a manifold structure, one has to impose certain asymptotic conditions at infinity for these diffeomorphisms. This amounts to working in weighted spaces. It is reasonable to expect that our results about the geodesic flow of the L^2 and H^1 right-invariant metric are valid in this setting as well. However, dealing with the arising weighted spaces is technically more cumbersome. Even if the study of the geodesic flow in this case is still quite incomplete, the rather formal association of the geodesic flow with solutions of (1.2) decaying at infinity is very useful in the study of the existence of permanent and breaking waves for the hydrodynamic model (1.2), see [12]. From

[18] The understanding is still incomplete but some significant results were obtained: in [40] it is proved that in three dimensions there exists a pair of volume-preserving diffeomorphisms that cannot be connected by a shortest path (*a priori* this does not rule out the possibility that the attractive variational approach works locally).

a qualitative point of view, perhaps the most interesting question is whether any two elements in $\mathcal{D}$ (or in the group of diffeomorphisms of the line), endowed with the H^1 right-invariant metric, can be joined by a geodesic and, if that is the case, whether the geodesic is length minimizing. □

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